



# **EXISTENCE RESULTS AND SOLUTION METHODS FOR CERTAIN VARIATIONAL INEQUALITIES AND COMPLEMENTARITY PROBLEMS**

THESIS SUBMITTED FOR THE DEGREE OF

**Doctor of Philosophy**

IN

**MATHEMATICS**

BY

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UNDER THE SUPERVISION OF

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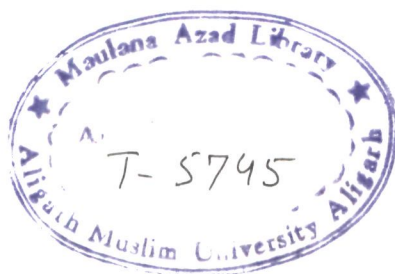
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Dated 30/12/2002

## Certificate

This is to certify that the thesis entitled "*Existence results and solution methods for certain variational inequalities and complementarity problems*" submitted for the award of the *Degree of Doctor of Philosophy* in Mathematics of Aligarh Muslim University, Aligarh embodies the original research work carried out by *Mr. Syed Shakaib Irfan* under my guidance and supervision and has not been submitted for the award of any other degree or diploma of this or any other University.

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# Preface

Mathematics is a central element of our current technology but few people realize that this celebrated high technology is so strongly based on Mathematics. The theory of variational inequalities is a powerful and elegant tool of the current mathematical technology and have become a rich source of inspiration for scientist and engineers. There are numerous standard textbooks and monographs dealing with various aspects of this domain. In the last four decades, this theory has been extended and generalized in various directions. Simultaneously, the theory of complementarity problems is grown up because of the applications to a wide class of mathematical models related to optimization, game theory, economics, engineering, mechanics, elasticity, fluid mechanics, stochastic optimal control, etc. For further detail on complementarity problems, we refer to a recent monograph by Isac [57] and references therein. There are three different aspects to study variational inequalities and complementarity problems (i) *Mathematical Modelling*: To convert the problems of real life or the problems from science, engineering and social sciences into a variational inequality problem/complementarity problem is called mathematical modelling. (ii) *Existence Theory*: To study the existence of solutions of variational inequalities/complementarity problems. (iii) *Numerical Methods*: To find the algorithms for computing the approximate solutions of variational inequalities/complementarity problems, which converge to the exact solution.

This thesis deals with existence theory and numerical methods of different kinds of variational inequalities, variational-like inequalities, variational inclusions and complementarity problems.

Chapter 1 deals with the brief introduction of variational inequalities, variational-like inequalities, variational inclusions and complementarity problems besides some basic definitions and results from the Functional Analysis.

In chapter 2, we consider the completely generalized nonlinear variational-like inequalities/inclusions with or without compact valued mappings. We first prove the existence of weak solutions of completely generalized nonlinear variational-like inequality problem in the setting of locally convex Hausdorff topological vector spaces. Secondly, we propose an iterative algorithm for computing the approximate solutions of completely generalized nonlinear variational-like inclusions with noncompact valued mappings in the setting of Hilbert spaces. We prove that the approximate solutions obtained by the proposed algorithm converge to the exact solution of our variational-like inclusion. We also prove that the existence of a solution of our problem. Some special cases are also discussed. In the last section of this chapter, we consider the random generalization of completely generalized nonlinear variational-like inclusions. The iterative algorithm for finding the approximate solutions, their convergence and existence of a solution are also discussed.

In chapter 3, we suggest the iterative methods for computing the approximate solutions of generalized nonlinear quasi-variational inclusion problems. The existence and convergence of solutions obtained by suggested algorithms are also studied. Several special cases are also mentioned.

In chapter 4, we consider two different classes of generalized co-quasi-variational inequalities in the setting of Banach spaces. By using the sunny nonexpensive retractions, we construct the projection iterative methods for finding the approximate solutions of our problems. Some existence and convergence results are also derived. In the last section, we consider the multivalued co-quasi-variational inequality problem for fuzzy mappings. Following the technique of the first two sections, we give an iterative algorithm and prove the convergence results for the approximate solutions obtained by proposed algorithm. The existence result for a solution of this problem is also investigated.



In chapter 5, we consider the generalized nonlinear variational inclusion problem (for short, GNVIP) in the setting of Banach spaces. Several special cases of (GNVIP) are also given. By using the resolvent operator technique for  $m$ -accretive operator defined on a Banach space, we convert our problem into a fixed point problem. This characterization is used to propose an iterative algorithm for computing the approximate solutions of (GNVIP). The convergence of approximate solutions obtained by the proposed algorithm and the existence of a solution of (GNVIP) are also studied. In the last section of this chapter, we extend the generalized nonlinear variational inclusion problem for fuzzy mappings. We also extend the iterative algorithm, and convergence and an existence result of second section of this chapter for fuzzy mappings.

Finally, in chapter 6, we study a class of generalized quasi-complementarity problems with fuzzy multivalued mappings and suggest a new algorithm for computing the approximate solutions of this class of generalized quasi-complementarity problems. We also discuss the existence of a solution of our problem without compactness assumption and the convergence of the iterative sequences generated by the algorithm. Some special cases are also given.

The section 3.2 and chapter 6 have been accepted for the publication in *Mathematical and Computational Applications* and *International Journal of Fuzzy Systems*, respectively.

# Acknowledgements

At the outset I bow before God whose blessings have always been with me and the same made it possible to complete this work in time.

It gives me immense pleasure to acknowledge and express my deep sense of gratitude to my supervisor Dr. Qamrul Hasan Ansari, Reader, Department of Mathematics, Aligarh Muslim University, Aligarh, who, inspite of his very busy schedule, has rendered all possible help to carry out my research work. It is his expert guidance, invaluable support and personal influence that enabled me to complete the work in the present form. Words fail me to express my thankfulness for his never failing inspiration.

I shall never forget to remember the sagacious advice of Dr. S. M. A. Zaidi, Chairman, Department of Mathematics, Aligarh Muslim University, Aligarh. I am also thankful to staff members who developed in me a sense of analytical approach through fruitful discussions.

I shall fail in my duty if I do not place on record my thanks to Prof. Abul Hasan Siddiqi, Department of Mathematical Sciences, King Fahd University of Petroleum and Minerals, Dhahran, Saudi Arabia, for his worthwhile suggestions throughout my research work.

I am also grateful to Dr. Rais Ahmad for providing me necessary help and support during completion and compilation of this work. Inspite of his hectic schedule he always spared his valuable time to guide me which is highly commendable and unforgettable.

Thanks also to my colleagues and friends especially Dr. Mohd. Firdaus Khan, Dr. Salahuddin and Mr. Zubair Khan for their valuable suggestions and support.

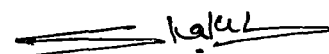
Note of thanks to all the family members who stood by me with their love, affection and understanding nature, which lighted up my spirit in hours of despair. I am particularly grateful to my parents, my father Dr. Qazi Mohd. Irfan and my mother

Mrs. Shahnaz Irfan for their patience, caring nature and help rendered to me in innumerable ways.

I am especially thankful to my sister Dr. Sheeba Saad and my brothers Syed Shueb, Syed Shariq and Saad Saeed who poured profound love and affection on me and have been a source of happiness to my life.

I am also indebted to my Angela niece Shifa who brought joy to my life.

Aligarh Muslim University, Aligarh,  
December 20, 2002

  
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# Chapter 1

## Preliminaries

In this chapter, we first present some basic definitions and results from Functional Analysis which will be used in the sequel. Then we give a brief introduction of variational inequalities, variational-like inequalities, variational inclusions and complementarity problems.

### 1.1 Introduction

*As long as a branch of knowledge offers an abundance of problems, it is full of vitality.*

*David Hilbert*

Because of the applications of Functional Analysis in Sciences, Engineering and Social Sciences, a great deal of work has been done in this area. Specially, the Nonlinear Analysis, a branch of Functional Analysis, has grown very rapidly and has many interesting applications in partial differential equations, mechanics, optimization, game theory, economics, engineering, etc. The theory of variational inequalities is one of the fields of applications of Nonlinear Analysis. It was introduced in sixties by the Italian and French Schools as a joint and concerted efforts of two leading mathematicians of this period, Guido Stampacchia and Jacques-Louis Lions. This theory has many applications in different branches of Sciences. In the last four decades, variational inequalities have been extended and generalized in different directions.

In the second section, we present some definitions and results from Functional Analysis which will be used in the sequel. The third section deals with the brief introduction of variational inequalities. Section 4 is devoted to the variational-like inequalities which are generalizations of variational inequalities. We also present different kinds of variational-like inequalities in this section. In section 5, we present the generalization of variational inequalities, known as variational inclusions. A brief introduction of complementarity problems is given in the last section of this chapter.

## 1.2 Tools from Functional Analysis

In this section, we present some basic definitions and results from Functional Analysis which will be used in the subsequent chapters.

Throughout the thesis, we shall use the following notations. For any nonempty set  $X$ , we denote by  $2^X$ ,  $CB(X)$  and  $C(X)$ , the family of all nonempty subsets of  $X$ , the family of all nonempty, closed and bounded subsets of  $X$ , and family of all nonempty compact subsets of  $X$ , respectively. If  $A$  is a nonempty subset of a topological vector space  $Y$  then we denote by  $\text{co}(A)$ ,  $\text{int}(A)$  and  $\partial\phi$ , the convex hull, the interior of  $A$  and the subdifferential of  $\phi$ , respectively.

Let  $H$  be a Hilbert space with its dual  $H^*$  whose norm and inner product are denoted by  $\|\cdot\|$  and  $\langle \cdot, \cdot \rangle$ , respectively.

**Definition 1.2.1.** A mapping  $g : H \rightarrow H$  is called:

- (i) *monotone* if  $\langle g(x) - g(y), x - y \rangle \geq 0$ ,  $\forall x, y \in H$ ; (1.2.1)
- (ii) *strictly monotone* if equality holds in (1.2.1) only if  $x = y$ ;
- (iii) *strongly monotone* if there exists a constant  $r > 0$  such that

$$\langle g(x) - g(y), x - y \rangle \geq r\|x - y\|^2, \quad \forall x, y \in H;$$

- (iv) *Lipschitz continuous* if there exists a constant  $s > 0$  such that

$$\|g(x) - g(y)\| \leq s\|x - y\|, \quad \forall x, y \in H.$$

**Theorem 1.2.1.** [86] *Let  $K$  be a nonempty, closed and convex subset of Hilbert space  $H$ . Then  $\forall z \in H, \exists$  unique  $u \in K$  such that*

$$\|z - u\| = \inf_{v \in K} \|z - v\|. \quad (1.2.2)$$

**Definition 1.2.2.** The point  $u$  satisfying (1.2.2) is called the *projection of  $z$  onto  $K$*  and we write

$$u = P_K(z).$$

**Lemma 1.2.2.** [68]. *If  $K$  is a nonempty, closed and convex subset of  $H$  and  $z$  is a given point in  $H$ , then  $u \in K$  satisfies the inequality*

$$\langle u - z, v - u \rangle \geq 0, \quad \forall v \in K,$$

*if and only if*

$$u = P_K(z), \quad (1.2.3)$$

*where  $P_K$  is the projection of  $H$  onto  $K$ .*

**Lemma 1.2.3.** [68]. *The mapping  $P_K$  defined by (1.2.3) is nonexpansive, i.e.,*

$$\|P_K(u) - P_K(v)\| \leq \|u - v\|, \quad \forall u, v \in H.$$

**Lemma 1.2.4.** [71]. *If  $K(u) = m(u) + K$  and  $K$  is a nonempty, closed and convex subset of  $H$ , then  $\forall u, v \in H$ ,*

$$P_{K(u)}v = m(u) + P_K(v - m(u)), \quad (1.2.4)$$

*where  $m : H \rightarrow H$  is a mapping.*

**Theorem 1.2.5.** [86] (*Riesz representation theorem*) *If  $f$  is bounded linear functional on a Hilbert space  $H$ , there exists a unique vector  $v \in H$  such that*

$$f(u) = \langle u, v \rangle, \quad \forall u \in H \text{ and } \|f\| = \|v\|.$$

Now, we present some basic definitions and results from the Set-Valued Analysis which will be used in the sequel.

**Definition 1.2.3.** Let  $X$  and  $Y$  be topological vector space. A multivalued mapping  $P : X \rightarrow 2^Y$  is called

- (i) *upper semicontinuous* at  $x_0 \in X$  if for every open set  $V$  in  $Y$  containing  $P(x_0)$ , there exists an open neighborhood  $U$  of  $x_0$  in  $X$  such that  $P(x) \subseteq V$ ,  $\forall x \in U$ ,
- (ii) *closed* if for every net  $\{x_\lambda\}$  converges to  $x_*$  and  $\{y_\lambda\}$  converges to  $y_*$  such that  $\forall \lambda, y_\lambda \in P(x_\lambda)$  implies that  $y_* \in P(x_*)$ ,
- (iii) *graph* of  $P$ , denoted by  $G(P)$  is

$$G(P) = \{(x, z) \in X \times Y : x \in X, z \in P(x)\}.$$

**Remark 1.2.1.** (i) Every upper semicontinuous multivalued map is closed.  
(ii) A multivalued map is closed if its graph is closed.

**Theorem 1.2.6.** [7] *Let  $K$  be a nonempty convex subset of a Hausdorff topological vector space  $X$ , and let  $P, Q : K \rightarrow 2^K$  be two multivalued maps. Assume that the following conditions hold.*

- (i) *For each  $x \in K$ ,  $co(P(x)) \subseteq Q(x)$  and  $P(x)$  is nonempty.*
- (ii)  *$K = \bigcup \{int_K P^{-1}(y) : y \in K\}$ .*
- (iii) *If  $K$  is not compact, assume that there exist a nonempty compact convex subset  $B$  of  $K$  and a nonempty compact subset  $D$  of  $K$  such that for each  $x \in K \setminus D$  there exists  $\tilde{y} \in B$  such that  $x \in int_K P^{-1}(\tilde{y})$ .*

*Then there exists  $\bar{x} \in K$  such that  $\bar{x} \in Q(\bar{x})$ .*

**Definition 1.2.4.** [27] A multivalued mapping  $Q : H \rightarrow 2^H$  is said to be  $\mathcal{H}$ -Lipschitz continuous if there exist a constant  $\gamma > 0$  such that

$$\mathcal{H}(Q(x), Q(y)) \leq \gamma \|x - y\|, \quad \forall x, y \in H,$$

where  $\mathcal{H}(\cdot, \cdot)$  is a Hausdorff metric on  $H$ .

**Definition 1.2.5.** A multivalued mapping  $Q : H \rightarrow 2^H$  is called:

- (i) *monotone* if  $\langle u - v, x - y \rangle \geq 0$ ,  $\forall x, y \in H, u \in Q(x), v \in Q(y)$ ; (1.2.5)
- (ii) *strictly monotone* if equality holds in (1.2.5) only if  $x = y$ ;
- (iii) *strongly monotone* if there exists a constant  $r > 0$  such that

$$\langle u - v, x - y \rangle \geq r \|x - y\|^2, \quad \forall x, y \in H, u \in Q(x), v \in Q(y);$$

(iv) *maximal monotone* if and only if  $Q$  is monotone and there is no other monotone mapping whose graph contains strictly the graph of  $Q$ .

**Lemma 1.2.7.** [12] *Let  $B$  be a reflexive Banach space endowed with strictly convex norm and  $\phi : B \rightarrow \mathbb{R} \cup \{+\infty\}$  be proper, convex, lower semicontinuous function. Then the subdifferential map  $\partial\phi : B \rightarrow 2^{B^*}$  is maximal monotone.*

**Definition 1.2.6.** [11, 105] Let  $Q : H \rightarrow 2^H$  be a maximal monotone mapping. For any fixed  $\alpha > 0$ , the mapping  $J_\alpha^Q(x) : H \rightarrow H$  defined by

$$J_\alpha^Q(x) = (I + \alpha Q)^{-1}(x), \quad \forall x \in H,$$

is called *resolvent operator* of  $Q$ , where  $I$  stands for the identity mapping on  $H$ .

The resolvent operator  $J_\alpha^Q$  is single valued and nonexpansive, that is,

$$\|J_\alpha^Q(x) - J_\alpha^Q(y)\| \leq \|x - y\| \quad \forall x, y \in H.$$

Since the subdifferential  $\partial\phi$  of a proper, convex, lower semicontinuous function  $\phi : H \rightarrow \mathbb{R} \cup \{+\infty\}$  is a maximal monotone multivalued map, it follows that the resolvent operator  $J_\alpha^{\partial\phi}$  of index  $\alpha$  of  $\partial\phi$  is given by

$$J_\alpha^{\partial\phi}(x) = (I + \alpha\partial\phi)^{-1}(x), \quad \forall x \in H.$$

**Definition 1.2.7.** A mapping  $J : B \rightarrow B^*$  is called *normalized duality mapping* if

$$\|J(x)\|_* = \|x\| \text{ and } \langle x, J(x) \rangle = \|x\|^2, \quad \forall x \in B.$$

The *uniform convexity* of the space  $B$  means that for any given  $\epsilon > 0$  there exists  $\delta > 0$  such that  $\forall x, y \in B, \|x\| \leq 1, \|y\| \leq 1, \|x - y\| = \epsilon$  ensure the following inequality

$$\|x + y\| \leq 2(1 - \delta).$$

The function

$$\delta_B(\epsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} : \|x\| = 1, \|y\| = 1, \|x - y\| = \epsilon \right\}$$

is called the *modulus of the convexity* of the space  $B$ .

The *uniform smoothness* of the space  $B$  means that for any given  $\epsilon > 0$ , there exists  $\delta > 0$  such that

$$\frac{\|x + y\| + \|x - y\|}{2} - 1 \leq \epsilon\|y\|$$

holds.



The function

$$\rho_B(t) = \sup \left\{ \frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| = 1, \|y\| = t \right\}$$

is called the *modulus of the smoothness* of space  $B$ .

*Remark 1.2.2.* The space  $B$  is uniformly convex if and only if  $\delta_B(\epsilon) > 0$  for all  $\epsilon > 0$ , and it is uniformly smooth if and only if  $\lim_{t \rightarrow 0} t^{-1} \rho_B(t) = 0$ .

**Proposition 1.2.8.** [2] Let  $B$  be a uniformly smooth Banach space and  $J$  be a normalized duality mapping from  $B$  to  $B^*$ . Then,  $\forall x, y \in B$ , we have

$$(i) \quad \|x+y\|^2 \leq \|x\|^2 + 2\langle y, J(x+y) \rangle,$$

$$(ii) \quad \langle x-y, J(x) - J(y) \rangle \leq 2d^2 \rho_B(4\|x-y\|/d),$$

where  $d = \sqrt{(\|x\|^2 + \|y\|^2)/2}$ .

**Definition 1.2.8.** The mapping  $G : B \rightarrow B$  is said to be *strongly accretive* if there exist a constant  $\gamma > 0$  such that

$$\langle G(x) - G(y), J(x-y) \rangle \geq \gamma \|x-y\|^2, \quad \forall x, y \in B.$$

**Definition 1.2.9.** [13, 39] A mapping  $Q_\Omega : B \rightarrow \Omega$ , where  $\Omega$  be a nonempty, closed and convex subset of  $B$ , is said to be

$$(i) \quad \text{retraction on } \Omega \text{ if } Q_\Omega^2 = Q_\Omega,$$

$$(ii) \quad \text{nonexpansive retraction if it satisfies the inequality}$$

$$\|Q_\Omega x - Q_\Omega y\| \leq \|x - y\|, \quad \forall x, y \in B,$$

$$(iii) \quad \text{sunny retraction if for all } x \in B \text{ and for all } -\infty < t < \infty$$

$$Q_\Omega(Q_\Omega x + t(x - Q_\Omega x)) = Q_\Omega x.$$

**Proposition 1.2.9.** [39]  $Q_\Omega$  is a sunny nonexpansive retraction if and only if  $\forall x, y \in B$

$$\langle x - Q_\Omega x, J(Q_\Omega x - y) \rangle \geq 0.$$

**Proposition 1.2.10.** [3] Let  $\Omega$  be a nonempty, closed and convex subset of a Banach space  $B$  and let  $m : B \rightarrow B$  be a mapping. Then  $\forall x \in B$ , we have

$$Q_{\Omega+m(x)}x = m(x) + Q_\Omega(x - m(x)).$$

**Definition 1.2.10.** [14] A multivalued map  $A : D(A) \subset B \rightarrow 2^B$  is said to be

(i) *accretive* if  $\forall x, y \in D(A), u \in A(x), v \in A(y), \exists j(x - y) \in J(x - y)$  such that

$$\langle u - v, j(x - y) \rangle \geq 0,$$

(ii) *k-strongly accretive*,  $k \in (0, 1)$  if  $\forall x, y \in D(A), \exists j(x - y) \in J(x - y)$  such that  $\forall u \in A(x), v \in A(y)$ ,

$$\langle u - v, j(x - y) \rangle \geq k\|x - y\|^2,$$

(iii) *m-accretive* if  $A$  is accretive and  $(I + \rho A)(D(A)) = B$  for every  $\rho > 0$ , where  $I$  is the identity mapping.

*Remark 1.2.3.* [29] If  $B = B^* = H$ , Hilbert space, then  $A : D(A) \subset H \rightarrow 2^H$  is an *m-accretive* mapping if and only if  $A : D(A) \subset H \rightarrow 2^H$  is a maximal monotone mapping.

## 1.3 Variational Inequalities

In this section, we present a brief introduction of variational inequalities.

The theory of variational inequalities was introduced in connection of a mechanical problem by Fichera [36], and Lions and Stampacchia [64]. Many problems of elasticity and fluid mechanics can be expressed in terms of an unknown  $u$ , representing the displacement of a mechanical system satisfying

$$a(u, v - u) \geq f(v - u), \quad \forall v \in K, \quad (1.3.1)$$

where the set  $K$  of admissible displacements is a nonempty, closed and convex subset of a Hilbert space  $H$ ,  $a(., .)$  is a bilinear form and  $f$  is a bounded linear functional on  $H$ . The inequalities of type (1.3.1) are called *variational inequalities*.

If the bilinear form  $a(., .)$  is continuous, then by Riesz-representation Theorem 1.2.6, we have

$$a(u, v) = \langle A(u), v \rangle, \quad \forall u, v \in H, \quad (1.3.2)$$

where  $A$  is a continuous linear operator on  $H$ . Then the inequality (1.3.1) is equivalent to find  $u \in K$  such that

$$\langle A(u), v - u \rangle \geq \langle f, v - u \rangle, \quad \forall v \in K. \quad (1.3.3)$$

If the operator  $A$  and  $f$  are nonlinear, the variational inequality (1.3.3) is known as *strongly nonlinear variational inequality*, introduced and studied by Noor [70].

If  $f \equiv 0$ , then (1.3.3) is equivalent to find  $u \in K$  such that

$$\langle A(u), v - u \rangle \geq 0, \quad \forall v \in K. \quad (1.3.4)$$

For further detail on variational inequalities, see [12, 24, 25, 37, 38, 57, 60, 83].

It is worth mentioning that the unilateral contact problem involving contact laws of monotone nature do not lead to the formulation of variational inequalities directly. However, it has been shown by Panagiotopoulos [80], using the notions of Clarke's generalized gradient and Rockafeller's upper subderivative, that the nonconvex unilateral contact problems can only be characterized by a class of strongly nonlinear variational inequalities (1.3.3).

If in a variational inequality formulation, the convex set  $K$  does depend upon its solution, then this class of variational inequalities is called a quasi-variational inequality. This class of inequalities is introduced and studied by Bensoussan, Goursat, and Lions [15]. For further detail, we refer to Bensoussan and Lions [16], Baiocchi and Capelo [12], and Mosco [68].

## 1.4 Variational-like Inequalities

The theory of variational inequalities has been extended and generalized in many different directions because of the applications in different branches of sciences, engineering, optimization, economics, equilibrium theory etc. The variational-like inequality is one of the generalized form of variational inequality, which was initially studied by Parida, Sahoo and Kumar [82] in 1989. They developed a theory for existence of its solutions in finite dimension Euclidean space  $\mathbb{R}^n$  by using Kakutani fixed point

theorem and they have also shown its relationship with mathematical programming. In 1992, Yang and Chen [99] have introduced a new class of nonconvex nonsmooth function (semi-preinvex functions). They derived the Fritz-John condition by using an alternative theorem for semi-preinvex program and studied the variational-like inequality and shown that it is a necessary condition for the optimal solutions. Some existence theorems are also proved. In 1994, Siddiqi, Khaliq and Ansari [92] studied variational-like inequalities in the setting of reflexive Banach spaces and topological vector spaces with or without convexity assumptions. Recently, Noor [74] has suggested an algorithm to find the approximate solutions of variational-like inequality and prove that the minimum of the arcwise directional differentiable semiconvex function [10] can be characterized by a class of variational-like inequalities. By using  $\eta$ -subdifferentiability, Lee, Ansari and Yao [63] suggested an perturbed iterative algorithm for finding the approximate solutions of variational-like inequalities. Dien [30] has studied a more general class of variational-like and quasivariational-like inequalities. Ansari and Yao [9] studied the existence of solutions of general variational-like inequalities and also proposed auxiliary principle technique to compute the approximate solutions.

In this section, we present different kinds of scalar-valued variational-like inequalities studied till now. These inequalities have not been much studied, therefore, there is a lot of scope to do research in this area.

Let  $\langle E, E^* \rangle$  be a dual system of locally convex spaces and  $K$  be a nonempty, closed and convex subset of  $E$ . Given two mappings  $f : E \rightarrow E^*$  and  $\eta(.,.) : K \times K \rightarrow E$ , then the *variational-like inequality problem* is the following:

$$(VLIP) \quad \begin{cases} \text{Find } x \in K \text{ such that} \\ \langle f(x), \eta(y, x) \rangle \geq 0, \quad \forall y \in K. \end{cases} \quad (1.4.1)$$

Inequality (1.4.1) is called *variational-like inequality*. It is first studied in 1989 by Parida, Sahoo and Kumar [82] in the setting of  $n$ -dimensional Euclidean space  $\mathbb{R}^n$  in the study of mathematical programming problem. It has been further studied by Yang and Chen [99] in the study of economic equilibrium problems, and Siddiqi, Khaliq and Ansari [92] in the setting of reflexive Banach spaces and topological vector spaces.

Recently, Noor [74], Ansari and Yao [9] have studied such type of problems by using the auxiliary variational principle technique and suggested an iterative algorithms.

Dien [30] has considered the following general variational-like inequality problem in  $\mathbb{R}^n$ . Given  $\phi : K \rightarrow \mathbb{R}$  find  $x \in K$  such that

$$(gVLIP) \quad \langle f(x), \eta(y, x) \rangle \geq \phi(x) - \phi(y), \quad \forall y \in K. \quad (1.4.2)$$

It has been further studied by Siddiqi, Ansari and Ahmad [90] in the setting of reflexive Banach spaces and topological vector spaces with or without convexity assumptions.

*Remark 1.4.1.* If  $\eta(y, x) = y - x$ , then variational-like inequality problem and general variational-like inequality problem become the following variational inequality problem.

In many applications, the convex set in the formulation of (VLIP) also depends upon the solution itself. In this case (VLIP) is called quasi-variational-like inequality problem. More precisely, for a given a multivalued map  $Q : K \rightarrow 2^K$ , the *general quasi-variational-like inequality problem* [30] is the following:

$$(GQVLIP) \quad \begin{cases} \text{Find } x \in K \text{ such that } x \in Q(x) \text{ and} \\ \langle f(x), \eta(y, x) \rangle \geq \phi(x) - \phi(y), \quad \forall y \in Q(x). \end{cases} \quad (1.4.3)$$

Let  $T : E \rightarrow 2^{E^*}$  be a multivalued map, then the *variational-like inequality problem for multivalued map* defined as follows:

$$\begin{cases} \text{Find } x \in K \text{ such that } u \in T(x) \text{ and} \\ \langle u, \eta(y, x) \rangle \geq 0, \quad \forall y \in K. \end{cases} \quad (1.4.4)$$

In the same way, we define *general variational-like inequality problem for multivalued maps* as follows:

$$\begin{cases} \text{Find } x \in K \text{ such that } u \in T(x) \text{ and} \\ \langle u, \eta(y, x) \rangle \geq \phi(x) - \phi(y), \quad \forall y \in K. \end{cases} \quad (1.4.5)$$

More general, we consider the *generalized variational-like inequality problem*. Let  $C$  be a nonempty subset of  $E^*$  and  $N : K \times C \rightarrow E^*$  and  $\eta : K \times K \rightarrow E$  be two single

valued mappings. Let  $T : K \rightarrow 2^C$  be a multivalued mapping. Then the *generalized variational-like inequality problem* is the following:

$$(GVLIP) \quad \begin{cases} \text{Find } x \in K, \text{ and } u \in T(x) \text{ such that} \\ \langle N(x, u), \eta(y, x) \rangle \geq 0, \quad \forall y \in K. \end{cases}$$

Such problem is studied by Parida and Sen [81] in  $\mathbb{R}^n$ . They have also shown its relationship with nonlinear programming problem and saddle point problem. It has been further studied by Yao [101] with applications in generalized complementarity problems see also Cubiotti and Yao [28]

Again, if the convex set  $K$  involved in the formulation of problem (1.4.4) depends upon the solution itself then we have the *generalized quasi-variational-like inequality problem* [102] as follows: Given a multivalued mapping  $Q : K \rightarrow 2^K$ ,

$$(GQVLIP) \quad \begin{cases} \text{Find } x \in Q(x), \text{ and } u \in T(x) \text{ such that} \\ \langle N(x, u), \eta(y, x) \rangle \geq 0, \quad \forall y \in Q(x). \end{cases}$$

It is first studied by Yao [102] with applications in economic equilibrium problem. It has been further studied by Tian [97] Kum [61].

The following *generalized quasi-variational-like inequality* is considered by Ben-El-Mechaiekh and Isac [67].

$$(GQVLIP)(I) \quad \begin{cases} \text{Find } x \in Q(x), u \in T(x) \text{ such that} \\ \langle N(x, u), \eta(y, x) \rangle \geq \phi(x) - \phi(y), \quad \forall y \in Q(x). \end{cases}$$

Now, we consider a more general form of generalized variational-like inequalities which includes all above mentioned variational-like inequalities as special case.

Given  $b(., .) : K \times K \rightarrow \mathbb{R}$ , which is not necessarily differentiable mapping. Then we have the following problem:

$$\begin{cases} \text{Find } x \in K, u \in T(x) \text{ such that} \\ \langle N(x, u), \eta(y, x) + b(x, y) - b(x, x) \rangle \geq 0, \quad \forall y \in K. \end{cases} \quad (1.4.6)$$

It is studied by Siddiqi Ansari and Khan [91].

## 1.5 Variational Inclusions and Quasi-variational Inclusions

A useful and important generalization of variational inequalities is a mixed type variational inequality containing nonlinear term. Due to the presence of the nonlinear term, the projection method can not be used to study the existence and algorithm of solutions for the mixed type variational inequalities. In 1994, Hassouni and Moudafi [41] used the resolvent operator technique for maximal monotone mappings to study mixed type variational inequalities with single valued mappings which are called *variational inclusions* and developed a perturbed algorithm for finding approximate solutions of mixed variational inequalities.

Let  $H$  be a real Hilbert space endowed with a norm  $\|\cdot\|$  and inner product  $\langle \cdot, \cdot \rangle$  and given continuous mappings  $T, g : H \rightarrow H$ , with  $\text{Img} \cap \text{dom } \partial\phi \neq \emptyset$ , consider the following problem:

$$(GSNVIP) \quad \begin{cases} \text{Find } u \in H \text{ such that} \\ g(u) \cap \text{dom } \partial\phi \neq \emptyset \\ \langle T(u) - A(u), v - g(u) \rangle \geq \phi(g(u)) - \phi(v), \forall v \in H. \end{cases}$$

In (GSNVIP),  $A$  is nonlinear continuous mapping on  $H$ ,  $\partial\phi$  denotes the subdifferential of proper, convex and lower semicontinuous function  $\phi : H \rightarrow \mathbb{R} \cup \{+\infty\}$ .

The class of variational inclusions considered in [41] is more general than the class of variational and quasi-variational inequalities studied by Noor [70, 72], Isac [55] and Siddiqi and Ansari [87, 88]. More precisely, with the choice  $\phi = \delta_K$  the indicator function of closed convex set  $K$ , the class of strongly nonlinear variational inequality problem given by

$$\langle T(u) - A(u), v - g(u) \rangle \geq 0, \quad \forall v \in K, \quad (1.5.1)$$

is recovered.

For the case when  $\phi(\cdot, \cdot) = \delta_K(\cdot - m(u)) = \delta_{K-m(u)}(\cdot)$  is single valued mapping on  $H$ , the problem (GSNVIP) reduces to *general strongly quasi-variational inequality problem* given by

$$\langle T(u) - A(u), v - g(u) \rangle \geq 0, \quad \forall v \in K. \quad (1.5.2)$$

where the set  $K(u)$  is equal to  $K + m(u)$ .

In 1998, Huang [48] introduced and studied the following class of important generalized set-valued implicit variational inclusion problems in a Hilbert space  $H$ .

Let  $M : H \rightarrow 2^H$  is a maximal monotone mapping and  $\text{Range}(g)$  and  $\text{dom } M \neq \emptyset$ , where  $g : H \rightarrow H$  is a single valued mapping. For given mapping  $f, p : H \rightarrow H$  and  $A, S : H \rightarrow CB(H)$ , find  $u \in H, w \in A(u)$ , and  $y \in S(u)$  such that  $g(u) \in \text{dom } M$  and

$$0 \in f(w) - p(y) + M(g(u)). \quad (1.5.3)$$

In 2000, Shim et al [85], introduce and study a new class of quasi-variational inclusion, which is called *generalized set-valued strongly nonlinear quasi-variational inclusion*.

Given set-valued mappings  $G, S, T : H \rightarrow 2^H$ , single valued mappings  $p, m : H \rightarrow H$ , and  $N : H \times H \rightarrow H$ . Suppose that  $M : H \rightarrow 2^H$  is a maximal monotone mapping.

Find  $u \in H, x \in S(u), y \in T(u)$  and  $z \in G(u)$  such that  $P(u) - m(z) \in \text{dom } M$ ,

$$0 \in N(x, y) + M(P(u) - m(z)). \quad (1.5.4)$$

## 1.6 Complementarity Problems

Due to the applications in area such as, optimization theory, structural engineering, mechanics, elasticity, lubrication theory, economics, equilibrium theory on networks, stochastic optimal control, etc., the complementarity problem is one of the interesting and important problems which was introduced by G. E. Lemke in 1964, but Cottle [26] and Cottle and Dantzig [27] formally defined the linear complementarity problem and called it the fundamental problem. In recent years, it has been generalized and extended to many different directions. The implicit complementarity problem which is one of the generalizations of complementarity problems, is introduced by Isac [53, 54]; See also [56, 57] and references therein. In 1987, Noor [72] considered and studied an important and useful generalization of complementarity problem which is called



*mildly nonlinear complementarity problem*. Siddiqi and Ansari [89] considered a new class of implicit complementarity problem and studied the existence of its solution. An iterative algorithm is also given to find the approximate solutions of the new problem and proved that the approximate solutions converge to the exact solution. They also mentioned several special cases.

Let  $H$  be a Hilbert spaces whose norm and inner product are denoted by  $\|\cdot\|$  and  $\langle \cdot, \cdot \rangle$ , respectively. Let  $K$  be a closed convex cone in  $H$  and  $K^*$  be the polar cone of  $K$  defined by

$$K^* = \{u \in H : \langle u, v \rangle \geq 0, \quad \forall v \in K\}.$$

Let  $T : H \rightarrow H^*$  be a given nonlinear operator. Then the problem of finding  $u \in K$  such that

$$T(u) \in K^* \text{ and } \langle u, T(u) \rangle = 0, \quad (1.6.1)$$

is known as *nonlinear complementarity problem* (for short, NCP).

Let  $G : K \rightarrow K$  be a nonlinear map. Isac [57] extended (NCP) to the following problem which is known as *implicit complementarity problem*:

$$(ICP) \quad \begin{cases} \text{Find } u \in K \text{ such that} \\ G(u) \in K, T(u) \in K^* \text{ and} \\ \langle G(u), T(u) \rangle = 0. \end{cases}$$

The class of (ICP) is considered and extensively studied by Isac; See for example, [57] and references therein.

Further (ICP) is generalized by Siddiqi and Ansari [89] in the following form.

If  $K \subseteq H$  and  $T, A : H \rightarrow H$  are nonlinear mappings and  $G : K \rightarrow H$  is a mapping, then the *strongly nonlinear implicit complementarity problem* is the following:

$$(SNICP) \quad \begin{cases} \text{Find } u \in K \text{ such that} \\ G(u) \in K, T(u) + A(u) \in K^* \text{ and} \\ \langle G(u), T(u) + A(u) \rangle = 0, \quad \forall v \in K. \end{cases}$$

By considering  $A$  is a multivalued mapping, Chang and Huang [21] considered and studied a complementarity problem, known as *generalized multivalued implicit*

*complementarity problem*, that is, given a subset  $K \subseteq H$  and  $T, G : K \rightarrow H$  are single valued mappings and  $A : K \rightarrow 2^H$  is a multivalued mapping. Then the problem of finding  $u \in K$ , and  $y \in A(u)$  such that

$$G(u) \in K, \quad T(u) + y \in K^*, \quad \langle G(u), T(u) + y \rangle = 0. \quad (1.6.2)$$

Of course, if  $A$  is a single valued map, then (1.6.2) reduces to (SNICP).

Recently, several kind of complementarity problems for fuzzy mappings are considered and studied by Chang [18], Chang and Huang [22], Noor [73] and Lee et al [62]. For a detail on Complementarity problems and their applications, we refer to Isac [57] and references therein.

## Chapter 2

# Completely Generalized Nonlinear Variational-like Inclusions

In this chapter, we consider the completely generalized nonlinear variational-like inequalities/inclusions with or without compact valued mappings. We first prove the existence of weak solutions of completely generalized nonlinear variational-like inequality problems in the setting of locally convex Hausdorff topological vector spaces. Secondly, we propose an iterative algorithm for computing the approximate solutions of completely generalized nonlinear variational-like inclusions with noncompact valued mappings in the setting of Hilbert spaces. We prove that the approximate solutions obtained by the proposed algorithm converge to the exact solution of our variational-like inclusion. We also prove that the existence of a solution of our problem. Some special cases are also discussed. In the last of this chapter, we consider the random generalization of completely generalized nonlinear variational-like inclusions. The iterative algorithm for finding the approximate solutions, their convergence and existence of a solution are also discussed.

### 2.1 Introduction and Formulations

The variational-like inequality, also known as pre-variational inequality, is one of the generalized form of variational inequality; See, for example, [6, 9, 63, 74, 82, 92, 99] and references therein. The variational-like inequality and generalized variational-like

inequality problems are the powerful tool to study the nonconvex optimization problems, and nonconvex and nondifferentiable optimization problems, respectively; See, for instant, [6, 8, 63, 82, 99] and references therein. The resolvent operator technique initially used by Hassouni and Moudafi [41] and Adly [1] to suggest an iterative algorithm for finding the approximate solutions of variational inequality problems. In most of the papers on variational-like inequalities/inclusions and their generalizations appeared in the literature, only variational principle technique is used to find the approximate solutions of these problems. Recently, Lee, Ansari and Yao [63] developed the proximal map technique, which is an extension of the resolvent operator technique, to propose an iterative algorithm for computing the approximate solutions of variational-like inequality problems. In the resolvent operator technique, it is necessary to assume that the functional involved in the formulation of variational inequality problems to be convex and lower semicontinuous, but in [63] these conditions are not required.

Let  $H$  be a real Hilbert space whose norm and inner product are denoted by  $\|\cdot\|$  and  $\langle \cdot, \cdot \rangle$ , respectively. Given single valued mappings  $f, g, p : H \rightarrow H$ , a bifunction  $\eta : H \times H \rightarrow H$ , and multivalued mappings  $M, S, T : H \rightarrow 2^H$ , we consider the following *completely generalized nonlinear variational-like inclusion problem*:

$$(CGNVLIP) \quad \begin{cases} \text{Find } x \in H, u \in M(x), v \in S(x), \text{ and } w \in T(x) \text{ such that} \\ x \in \text{dom } \phi \text{ and} \\ \langle p(u) - (f(v) - g(w)), \eta(y, x) \rangle \geq \phi(x) - \phi(y), \quad \forall y \in H, \end{cases}$$

where  $\phi : H \rightarrow \mathbb{R} \cup \{+\infty\}$  and  $\text{dom } \phi = \{x \in H : \phi(x) < \infty\}$ .

**SPECIAL CASES:**

(i) If  $p \equiv 0$ ,  $f$ ,  $g$  and  $M$  are identity mappings, and  $S$  and  $T$  are single valued mappings, then (CGNVLIP) reduces to the problem of finding  $x \in H$  such that  $x \in \text{dom } \phi$  and

$$\langle T(x) - S(x), \eta(y, x) \rangle \geq \phi(x) - \phi(y), \quad \forall y \in H. \quad (2.1.1)$$

It is considered and studied by Lee, Ansari and Yao [63].

(ii) If  $\phi \equiv \delta_K$ , the indicator function of the closed convex set  $K$  in  $H$  defined by

$$\delta_K(x) = \begin{cases} 0, & x \in K \\ +\infty, & \text{otherwise,} \end{cases}$$

$f, g$  and  $M$  are identity mappings and  $\eta(y, x) = y - p(x)$ , then (CGNVLIP) becomes the problem of finding  $x \in K$ ,  $v \in S(x)$ , and  $w \in T(x)$  such that

$$\langle p(x) - (v - w), y - p(x) \rangle \geq 0, \quad \forall y \in K. \quad (2.1.2)$$

Such a problem is considered and studied by Verma [98]. (CGNVLIP) also contains many other kinds of variational inequalities and variational-like inequalities and inclusions studied in [63, 98] and references therein.

## 2.2 Existence Theory in Topological Vector Spaces

In this section, we consider the weak formulation of completely generalized nonlinear variational-like inequality problem in the setting of locally convex topological vector spaces and prove the existence of its solution.

Let  $\langle E, E^* \rangle$  be a dual system of locally convex spaces and  $K$  be a nonempty convex subset of  $E$ . Given single valued mappings  $f, g, p : E^* \rightarrow E^*$ , a bifunction  $\eta : K \times K \rightarrow E$ , multivalued maps  $M, S, T : K \rightarrow E^*$ , and a functional  $\phi : K \rightarrow \mathbb{R}$ , we consider the following *weak formulation of completely generalized nonlinear variational-like inequality problem*:

$$(WCGNVLIP) \quad \begin{cases} \text{Find } x \in K \text{ such that } \forall y \in K, \\ \exists u \in M(x), v \in S(x), \text{ and } w \in T(x) \text{ satisfying} \\ \langle p(u) - (f(v) - g(w)), \eta(y, x) \rangle \geq \phi(x) - \phi(y). \end{cases}$$

The solution of (WCGNVLIP) is called the *weak solution* of (CGNVLIP). We notice that  $u$ ,  $v$  and  $w$  depend on  $y$  and therefore the above mentioned problem is called the weak formulation of completely generalized nonlinear variational-like inequality problem. While every solution of completely generalized nonlinear variational-like inequality problem is a weak solution but the converse is not true in general.

We also remark that the existence of a weak solution of a generalized nonlinear variational-like inequality problem implies the existence of a nondifferentiable and nonconvex optimization problem under certain conditions; See for example [5, 8] and references therein.

If  $f$ ,  $g$  and  $M$  are identity mappings,  $\phi \equiv 0$  and  $\eta(y, x) = y - q(x)$ , where  $q : K \rightarrow K$  is a nonlinear operator, then (WCGNVLIP) reduces to the problem of finding  $x \in K$  such that  $\forall y \in K, \exists v \in S(x)$  and  $w \in T(x)$  satisfying

$$\langle p(x) - (v - w), y - q(x) \rangle \geq 0. \quad (2.2.1)$$

Throughout this section, we assume that the pairing  $\langle \cdot, \cdot \rangle$  is continuous.

**Theorem 2.2.1.** *Let  $E$  be a locally convex Hausdorff topological vector space and  $K$  be a nonempty convex subset of  $E$ . Let  $M, S, T : K \rightarrow 2^{E^*}$  be upper semicontinuous multivalued maps with nonempty and compact values and  $f, g, p : E^* \rightarrow E^*$  be continuous. Assume that the following conditions hold.*

- (i)  $\phi : K \rightarrow \mathbb{R}$  is continuous on  $K$ ;
- (ii)  $\eta : K \times K \rightarrow E$  is continuous in the second argument such that  $\eta(x, x) = 0, \forall x \in K$ ;
- (iii) The set  $\{y \in K : \forall u \in M(x), v \in S(x) \text{ and } w \in T(x) \text{ s.t. } \langle p(u) - (f(v) - g(w)), \eta(y, x) \rangle < \phi(x) - \phi(y)\}$  is convex,  $\forall x \in K$ ;
- (iv) If  $K$  is not compact, assume that there exist a nonempty compact convex subset  $B$  of  $K$  and a nonempty compact subset  $D$  of  $K$  such that  $\forall x \in K \setminus D, \exists \tilde{y} \in B$  satisfying  $\langle p(u) - (f(v) - g(w)), \eta(\tilde{y}, x) \rangle < \phi(x) - \phi(\tilde{y}), \forall u \in M(x), v \in S(x)$  and  $w \in T(x)$ .

Then (WCGNVLIP) has a solution.

*Proof.* Assume that the conclusion of this theorem does not hold. Then  $\forall x \in K$ , the set

$$\{y \in K : \forall u \in M(x), v \in S(x) \text{ and } w \in T(x) \text{ satisfying}$$

$$\langle p(u) - (f(v) - g(w)), \eta(y, x) \rangle < \phi(x) - \phi(y) \} \neq \emptyset.$$

We define a multivalued map  $Q : K \rightarrow 2^K$  by

$$Q(x) = \{y \in K : \forall u \in M(x), v \in S(x) \text{ and } w \in T(x) \text{ satisfying}$$

$$\langle p(u) - (f(v) - g(w)), \eta(y, x) \rangle < \phi(x) - \phi(y)\}, \quad \forall x \in K.$$

Then clearly  $Q(x) \neq \emptyset, \forall x \in K$ .

We prove that  $Q^{-1}(y)$  is open in  $K$ . For that it is sufficient to show that  $[Q^{-1}(y)]^c$ , the complement of  $Q^{-1}(y)$  in  $K$ , is closed in  $K$ .

Let  $\{x_\lambda\}_{\lambda \in \Lambda}$  be a net in  $[Q^{-1}(y)]^c$  such that  $\{x_\lambda\}$  converges to  $x^* \in K$ . Then  $\exists u_\lambda \in M(x_\lambda), v_\lambda \in S(x_\lambda)$  and  $w_\lambda \in T(x_\lambda)$  such that

$$\langle p(u_\lambda) - (f(v_\lambda) - g(w_\lambda)), \eta(y, x_\lambda) \rangle - \phi(x_\lambda) + \phi(y) \geq 0, \quad \forall \lambda.$$

Let  $\mathcal{A} = \{x_\lambda\} \cup \{x^*\}$ . Then  $\mathcal{A}$  is compact and  $u_\lambda \in M(\mathcal{A}), v_\lambda \in S(\mathcal{A})$  and  $w_\lambda \in T(\mathcal{A})$  which are compact. Therefore  $\{u_\lambda\}, \{v_\lambda\}$  and  $\{w_\lambda\}$  have convergent subnet with limit, say,  $u_*, v_*$  and  $w_*$ , respectively. Without loss of generality we may assume that  $\{u_\lambda\}$  converges to  $u_*$ ,  $\{v_\lambda\}$  converges to  $v_*$ , and  $\{w_\lambda\}$  converges to  $w_*$ . Then by the upper semicontinuity of  $M, S$  and  $T$ , we have  $u_* \in M(x^*), v_* \in S(x^*)$  and  $w_* \in T(x^*)$ . By the continuity of  $f, g, p$  and (i) and (ii), we have

$$\langle p(u_\lambda) - (f(v_\lambda) - g(w_\lambda)), \eta(y, x_\lambda) \rangle - \phi(x_\lambda) + \phi(y)$$

converges to

$$\langle p(u_*) - (f(v_*) - g(w_*)), \eta(y, x_*) \rangle - \phi(x_*) + \phi(y).$$

Hence

$$\langle p(u_*) - (f(v_*) - g(w_*)), \eta(y, x_*) \rangle \geq \phi(x_*) - \phi(y),$$

and therefore  $x^* \in [Q^{-1}(y)]^c$ . Thus  $[Q^{-1}(y)]^c$  is closed in  $K$ .

Since  $Q(x), \forall x \in K$  is nonempty and open in  $K$ , we have

$$K = \bigcup_{y \in K} Q^{-1}(y) = \bigcup_{y \in K} \text{int}_K Q^{-1}(y).$$

By (iii),  $Q(x), \forall x \in K$  is convex. Since  $\forall x \in K \setminus D, \exists \tilde{y} \in B$  satisfying  $\langle p(u) - (f(v) - g(w)), \eta(\tilde{y}, x) \rangle < \phi(x) - \phi(\tilde{y}), \forall u \in M(x), v \in S(x)$  and  $w \in T(x)$ , we have  $x \in \text{int}_K Q^{-1}(\tilde{y})$ . Hence  $Q$  satisfies all the conditions of Theorem 2.2.1. Therefore

by Theorem 1.2.6 with  $P \equiv Q$ ,  $\exists \bar{x} \in K$  such that  $\bar{x} \in Q(\bar{x})$ , that is,  $\forall \bar{u} \in M(\bar{x})$ ,  $\bar{v} \in S(\bar{x})$  and  $\bar{w} \in T(\bar{x})$ , we have

$$\langle p(\bar{u}) - (f(\bar{v}) - g(\bar{w})), \eta(\bar{x}, \bar{x}) \rangle < \phi(\bar{x}) - \phi(\bar{x}).$$

Since  $\eta(x, x) = 0, \forall x \in K$ , we reach to a contradiction. This completes the proof.  $\square$

**Corollary 2.2.2.** *Let  $E$  be a locally convex Hausdorff topological vector space and  $K$  be a nonempty convex subset of  $E$ . Let  $M, S, T : K \rightarrow 2^{E^*}$  be upper semicontinuous multivalued maps with nonempty and compact values and  $f, g, p : E^* \rightarrow E^*$  and  $\eta : K \times K \rightarrow E$  be functions such that  $\liminf_{x \rightarrow x^*, u \rightarrow u^*, v \rightarrow v^*, w \rightarrow w^*} \langle p(u) - (f(v) - g(w)), \eta(y, x) \rangle \leq \langle p(u^*) - (f(v^*) - g(w^*)), \eta(y, x^*) \rangle$ . Also, let  $\phi : K \rightarrow \mathbb{R}$  be continuous on  $K$  and the set  $\{y \in K : \forall u \in M(x), v \in S(x) \text{ and } w \in T(x) \text{ s.t. } \langle p(u) - (f(v) - g(w)), \eta(y, x) \rangle < \phi(x) - \phi(y)\}$  is convex,  $\forall x \in K$ . If  $K$  is not compact, assume that there exist a nonempty compact convex subset  $B$  of  $K$  and a nonempty compact subset  $D$  of  $K$  such that  $\forall x \in K \setminus D, \exists \tilde{y} \in B$  satisfying  $\langle p(u) - (f(v) - g(w)), \eta(\tilde{y}, x) \rangle < \phi(x) - \phi(\tilde{y}), \forall u \in M(x), v \in S(x) \text{ and } w \in T(x)$ . Then (WCGNVLIP) has a solution.*

## 2.3 Iterative Algorithms and Convergence Results

In this section, we consider the completely generalized nonlinear variational-like inclusions with noncompact valued mappings which include many known variational-like inclusions and variational inclusions as special cases. We propose an iterative algorithm for computing the approximate solutions of this class of variational-like inclusions. We prove that the approximate solutions obtained by the proposed algorithm converge to the exact solution of our variational-like inclusion. The existence of a solution of our problem is also studied.

Recently, Lee, Ansari and Yao [63] introduced the following concept of  $\eta$ -subdifferential in a more general setting than that given in [98].



Let  $\eta : H \times H \rightarrow H$  and  $\phi : H \rightarrow \mathbb{R} \cup \{+\infty\}$ . A vector  $v \in H$  is called an  $\eta$ -subgradient of  $\phi$  at  $x \in \text{dom } \phi$  if

$$\langle v, \eta(y, x) \rangle \leq \phi(y) - \phi(x), \quad \forall y \in H. \quad (2.3.1)$$

Each  $\phi$  can be associated with the following  $\eta$ -subdifferential map  $\partial_\eta \phi$  defined by

$$\partial_\eta \phi(x) = \begin{cases} \{v \in H : \langle v, \eta(y, x) \rangle \leq \phi(y) - \phi(x), \forall y \in H\}, & x \in \text{dom } \phi \\ \emptyset, & x \notin \text{dom } \phi. \end{cases}$$

For  $x \in \text{dom } \phi$ ,  $\partial_\eta \phi(x)$  is called  $\eta$ -subdifferential of  $\phi$  at  $x$ .

We require the following definitions to achieve the main result of this section.

**Definition 2.3.1.** [63] A mapping  $\eta : H \times H \rightarrow H$  is called:

(i) *monotone* if

$$\langle x - y, \eta(x, y) \rangle \geq 0, \quad \forall x, y \in H; \quad (2.3.2)$$

(ii) *strictly monotone* if the equality holds in (2.3.2) only when  $x = y$ ;

(iii) *strongly monotone* if there exists a constant  $\sigma > 0$  such that

$$\langle x - y, \eta(x, y) \rangle \geq \sigma \|x - y\|^2, \quad \forall x, y \in H;$$

(iv) *Lipschitz continuous* if there exists a constant  $\delta > 0$  such that

$$\|\eta(x, y)\| \leq \delta \|x - y\|, \quad \forall x, y \in H.$$

**Definition 2.3.2.** [98] Let  $f : H \rightarrow H$  be a mapping. A multivalued mapping  $Q : H \rightarrow 2^H$  is said to be:

(i) *relaxed Lipschitz with respect to  $f$*  if there exists a constant  $k \geq 0$  such that

$$\langle f(u) - f(v), x - y \rangle \leq -k \|x - y\|^2, \quad \forall x, y \in H, u \in Q(x), v \in Q(y);$$

(ii) *relaxed monotone with respect to  $f$*  if there exists a constant  $c > 0$  such that

$$\langle f(u) - f(v), x - y \rangle \geq -c \|x - y\|^2, \quad \forall x, y \in H, u \in Q(x), v \in Q(y).$$

**Definition 2.3.3.** Let  $\eta : H \times H \rightarrow H$  be a given map. A multivalued map  $Q : H \rightarrow 2^H$  is called  $\eta$ -monotone if for all  $x, y \in H$ ,

$$\langle u - v, \eta(x, y) \rangle \geq 0, \quad \forall u \in Q(x), v \in Q(y).$$

$Q$  is called *maximal  $\eta$ -monotone* if and only if it is  $\eta$ -monotone and there is no other  $\eta$ -monotone multivalued map whose graph strictly contains the graph of  $Q$ .

**Assumption 2.3.1.** The operator  $\eta : H \times H \rightarrow H$  satisfies the condition

$$\eta(y, x) + \eta(x, y) = 0, \quad \forall x, y \in H.$$

*Remark 2.3.1.* If  $\eta : H \times H \rightarrow H$  satisfies Assumption 2.3.1 and  $\phi : H \rightarrow \mathbb{R} \cup \{+\infty\}$ , then it is easy to see that the multivalued map  $\partial_\eta \phi : H \rightarrow 2^H$  is  $\eta$ -monotone.

We need the following result due to Lee, Ansari and Yao [63], to transform (CGN-VLIP) into a fixed point problem.

**Proposition 2.3.1.** [63] Let  $\eta : H \times H \rightarrow H$  be strictly monotone and  $Q : H \rightarrow 2^H$  be an  $\eta$ -monotone multivalued map. If, the range of  $(I + \lambda Q)$ ,  $R(I + \lambda Q) = H$ , for  $\lambda > 0$  and  $I$  is the identity operator, then  $Q$  is maximal  $\eta$ -monotone. Furthermore, the inverse operator  $(I + \lambda Q)^{-1}$  is single-valued.

Throughout this section, we will assume that  $\eta : H \times H \rightarrow H$  is strictly monotone and satisfies Assumption 2.3.1 and  $\phi : H \rightarrow \mathbb{R} \cup \{+\infty\}$  is a map such that  $R(I + \lambda \partial_\eta \phi) = H$  for  $\lambda > 0$ .

From Proposition 2.3.1, we note that the mapping

$$J_\lambda^\phi(x) = (I + \lambda \partial_\eta \phi)^{-1}(x), \quad \forall x \in H,$$

is single valued.

Let us transform (CGNLVLIP) into a fixed point problem.

**Lemma 2.3.2.** The set of elements  $(x, u, v, w)$  such that  $x \in H, u \in M(x), v \in S(x)$  and  $w \in T(x)$ , is a solution of (CGNLVLIP) if and only if it satisfies the following relation:

$$x = J_\lambda^\phi[x - \lambda(p(u) - (f(v) - g(w)))], \quad (2.3.3)$$

where  $\lambda > 0$  is a constant,  $J_\lambda^\phi = (I + \lambda \partial_\eta \phi)^{-1}$  is so-called proximal map and  $I$  stands for the identity operator on  $H$ .

*Proof.* From the definition of  $J_\lambda^\phi$ , we have

$$x - \lambda(p(u) - (f(v) - g(w))) \in x + \lambda \partial_\eta \phi(x)$$

and hence

$$(f(v) - g(w)) - p(u) \in \partial_\eta \phi(x).$$

By using the definition of  $\eta$ -subdifferential, we have

$$\langle (f(v) - g(w)) - p(u), \eta(y, x) \rangle \leq \phi(y) - \phi(x), \quad \forall y \in H.$$

Thus  $(x, u, v, w)$  is a solution of (CGNVLIP).  $\square$

From Lemma 2.3.2, we see that (CGNVLIP) is equivalent to the fixed point problem of type

$$x = q(x) \tag{2.3.4}$$

where  $q(x) = J_\lambda^\phi[x - \lambda(p(u) - (f(v) - g(w)))]$ ,

which implies that

$$x = (1 - \rho)x + \rho J_\lambda^\phi[x - \lambda(p(u) - (f(v) - g(w)))],$$

where  $0 < \rho < 1$  is a parameter and  $\lambda > 0$  is a constant.

On the basis of the above mentioned observations, we propose the following iterative algorithm to compute the approximate solutions of (CGNVLIP).

**Algorithm 2.3.1.** Assume that  $\eta : H \times H \rightarrow H$  is a map and  $f, g, p : H \rightarrow H$  are single valued mappings. Let  $S, M, T : H \rightarrow CB(H)$  be multivalued mappings. For given  $x_0 \in H$ , take  $u_0 \in M(x_0)$ ,  $v_0 \in S(x_0)$  and  $w_0 \in T(x_0)$ , and let

$$x_1 = (1 - \rho)x_0 + \rho J_\lambda^{\phi_0}[x_0 - \lambda(p(u_0) - (f(v_0) - g(w_0)))].$$

Since  $u_0 \in M(x_0) \in CB(H)$ ,  $v_0 \in S(x_0) \in CB(H)$ ,  $w_0 \in T(x_0) \in CB(H)$ , by Nadler [69], there exist  $u_1 \in M(x_1)$ ,  $v_1 \in S(x_1)$ ,  $w_1 \in T(x_1)$  such that

$$\begin{aligned}\|u_0 - u_1\| &\leq (1 + 1) \mathcal{H}(M(x_0), M(x_1)), \\ \|v_0 - v_1\| &\leq (1 + 1) \mathcal{H}(S(x_0), S(x_1)), \\ \|w_0 - w_1\| &\leq (1 + 1) \mathcal{H}(T(x_0), T(x_1)).\end{aligned}$$

Let

$$x_2 = (1 - \rho)x_1 + \rho J_\lambda^{\phi_1}[x_1 - \lambda(p(u_1) - (f(v_1) - g(w_1)))].$$

By induction, we can obtain sequences  $\{x_n\}$ ,  $\{u_n\}$ ,  $\{v_n\}$  and  $\{w_n\}$  as

$$x_{n+1} = (1 - \rho)x_n + \rho J_\lambda^{\phi_n}[x_n - \lambda(p(u_n) - (f(v_n) - g(w_n)))], \quad (2.3.5)$$

$$u_n \in M(x_n), \quad \|u_n - u_{n+1}\| \leq (1 + (n + 1)^{-1}) \mathcal{H}(M(x_n), M(x_{n+1})),$$

$$v_n \in S(x_n), \quad \|v_n - v_{n+1}\| \leq (1 + (n + 1)^{-1}) \mathcal{H}(S(x_n), S(x_{n+1})),$$

$$w_n \in T(x_n), \quad \|w_n - w_{n+1}\| \leq (1 + (n + 1)^{-1}) \mathcal{H}(T(x_n), T(x_{n+1})),$$

$n = 0, 1, 2, 3, \dots$

We need the following lemma due to Lee, Ansari and Yao [63].

**Lemma 2.3.3.** [63] *Let  $\eta : H \times H \rightarrow H$  be strongly monotone and Lipschitz continuous with constants  $t > 0$  and  $s > 0$ , respectively, and satisfy Assumption 2.3.1. Then*

$$\|J_\lambda^\phi(x) - J_\lambda^\phi(y)\| \leq \tau \|x - y\|, \quad \forall x, y \in H,$$

where  $\tau = \frac{s}{t}$ .

Now we are ready to prove the main result of this section.

**Theorem 2.3.4.** *Let  $\eta : H \times H \rightarrow H$  be strongly monotone and Lipschitz continuous with constants  $t > 0$  and  $s > 0$ , respectively, and satisfy Assumption 2.3.1. Let  $f, g, p : H \rightarrow H$  be Lipschitz continuous with corresponding constants  $\xi, r$  and  $\sigma$ , respectively. Let  $M, S, T : H \rightarrow CB(H)$  be  $\mathcal{H}$ -Lipschitz continuous with corresponding*

constants  $\gamma, h$  and  $d$ , respectively, and  $S$  be relaxed Lipschitz with respect to  $f$  with constant  $k$  and  $T$  be relaxed monotone with respect to  $g$  with constant  $c$ . For each  $n$ , let  $\phi_n : H \rightarrow \mathbb{R} \cup \{+\infty\}$  and  $\phi : H \rightarrow \mathbb{R} \cup \{+\infty\}$  be mappings such that  $R(I + \lambda \partial_\eta \phi_n) = R(I + \lambda \partial_\eta \phi) = H$  for  $\lambda > 0$ . Assume that

$$\lim_{n \rightarrow +\infty} \|J_\lambda^{\phi_n}(z) - J_\lambda^{\phi_{n-1}}(z)\| = 0, \quad \forall z \in H$$

and if

$$\left| \lambda - \frac{(k - c) - a\sigma\nu}{(\xi h + rd)^2 - \sigma^2\nu^2} \right| < \frac{\sqrt{(a\sigma\nu - (k - c))^2 - ((\xi h + rd)^2 - \sigma^2\nu^2)(1 - a^2)}}{(\xi h + rd)^2 - \sigma^2\nu^2},$$

$$(a\sigma\nu - (k - c)) > \sqrt{((\xi h + rd)^2 - \sigma^2\nu^2)(1 - a^2)}, \quad (2.3.6)$$

$$\xi h + rd > \sigma\nu, s > t, \sigma\nu > 1 \text{ and } t > 1,$$

where  $a = \frac{1}{\tau}$ .

Then, there exists a set  $(x, u, v, w)$  such that  $x \in H, u \in M(x), v \in S(x)$  and  $w \in T(x)$ , which is a solution of (CGNVLIP), and  $x_n \rightarrow x, u_n \rightarrow u, v_n \rightarrow v, w_n \rightarrow w$  as  $n \rightarrow \infty$ , where  $\{x_n\}, \{u_n\}, \{v_n\}$  and  $\{w_n\}$  are the sequences obtained by Algorithm 2.3.1.

*Proof.* From (2.3.5), we have

$$\|x_{n+1} - x_n\| = \|(1 - \rho)(x_n - x_{n-1}) + \rho J_\lambda^{\phi_n}(h(x_n)) - \rho J_\lambda^{\phi_{n-1}}(h(x_{n-1}))\|, \quad (2.3.7)$$

where  $h(x_n) = x_n - \lambda(p(u_n) - (f(v_n) - g(w_n)))$ .

By introducing the term  $J_\lambda^{\phi_n}(h(x_{n-1}))$ , we get

$$\begin{aligned} \|J_\lambda^{\phi_n}(h(x_n)) - J_\lambda^{\phi_{n-1}}(h(x_{n-1}))\| &\leq \|J_\lambda^{\phi_n}(h(x_n)) - J_\lambda^{\phi_n}(h(x_{n-1}))\| \\ &\quad + \|J_\lambda^{\phi_n}(h(x_{n-1})) - J_\lambda^{\phi_{n-1}}(h(x_{n-1}))\|. \end{aligned} \quad (2.3.8)$$

By Lemma 2.3.2, we have

$$\begin{aligned} \|J_\lambda^{\phi_n}(h(x_n)) - J_\lambda^{\phi_{n-1}}(h(x_{n-1}))\| &\leq \tau \|h(x_n) - h(x_{n-1})\| \\ &\quad + \|J_\lambda^{\phi_n}(h(x_{n-1})) - J_\lambda^{\phi_{n-1}}(h(x_{n-1}))\|, \end{aligned} \quad (2.3.9)$$

where  $\tau = \frac{s}{t}$ , and

$$\begin{aligned}
\|h(x_n) - h(x_{n-1})\| &= \|x_n - \lambda(p(u_n) - (f(v_n) - g(w_n))) \\
&\quad - x_{n-1} + \lambda(p(u_{n-1}) - (f(v_{n-1}) - g(w_{n-1})))\| \\
&\leq \|x_n - x_{n-1} + \lambda(f(v_n) - f(v_{n-1})) - \lambda(g(w_n) - g(w_{n-1}))\| \\
&\quad + \lambda\tau\|p(u_n) - p(u_{n-1})\|. \tag{2.3.10}
\end{aligned}$$

From (2.3.7) - (2.3.10), we get

$$\begin{aligned}
\|x_{n+1} - x_n\| &\leq (1 - \rho)\|x_n - x_{n-1}\| \\
&\quad + \rho\tau\|x_n - x_{n-1} + \lambda(f(v_n) - f(v_{n-1})) - \lambda(g(w_n) - g(w_{n-1}))\| \\
&\quad + \lambda\rho\tau\|p(u_n) - p(u_{n-1})\| \\
&\quad + \rho\|J_\lambda^{\phi_n}(h(x_{n-1})) - J_\lambda^{\phi_{n-1}}(h(x_{n-1}))\|. \tag{2.3.11}
\end{aligned}$$

Since  $M, S$  and  $T$  are  $\mathcal{H}$ -Lipschitz continuous, and  $f, g$  and  $p$  are Lipschitz continuous, we have

$$\|p(u_n) - p(u_{n-1})\| \leq \sigma\|u_n - u_{n-1}\| \leq \sigma\gamma(1 + 1/n)\|x_n - x_{n-1}\|, \tag{2.3.12}$$

$$\|f(v_n) - f(v_{n-1})\| \leq \xi\|v_n - v_{n-1}\| \leq \xi h(1 + 1/n)\|x_n - x_{n-1}\|, \tag{2.3.13}$$

$$\|g(w_n) - g(w_{n-1})\| \leq r\|w_n - w_{n-1}\| \leq rd(1 + 1/n)\|x_n - x_{n-1}\|. \tag{2.3.14}$$

Further, since  $S$  is relaxed Lipschitz and  $T$  is relaxed monotone, we have

$$\begin{aligned}
&\|x_n - x_{n-1} + \lambda(f(v_n) - f(v_{n-1})) - \lambda(g(w_n) - g(w_{n-1}))\|^2 \\
&= \|x_n - x_{n-1}\|^2 + 2\lambda\langle f(v_n) - f(v_{n-1}), x_n - x_{n-1} \rangle \\
&\quad - 2\lambda\langle g(w_n) - g(w_{n-1}), x_n - x_{n-1} \rangle + \lambda^2\|f(v_n) - f(v_{n-1}) - (g(w_n) - g(w_{n-1}))\|^2 \\
&\leq [1 - 2\lambda(k - c) + \lambda^2(1 + 1/n)^2(\xi h + rd)^2]\|x_n - x_{n-1}\|^2. \tag{2.3.15}
\end{aligned}$$

From (2.3.11) - (2.3.15), it follows that

$$\|x_{n+1} - x_n\| \leq \theta_n\|x_n - x_{n-1}\| + \rho\|J_\lambda^{\phi_n}(h(x_{n-1})) - J_\lambda^{\phi_{n-1}}(h(x_{n-1}))\|, \tag{2.3.16}$$

where  $\theta_n = (1 - \rho) + \rho[\tau\sqrt{(1 - 2\lambda(k - c) + \lambda^2(1 + \frac{1}{n})^2(\xi h + rd)^2)} + \lambda\sigma\gamma\tau(1 + \frac{1}{n})]$ .

Let  $\theta = (1 - \rho) + \rho[\tau\sqrt{(1 - 2\lambda(k - c) + \lambda^2(\xi h + rd)^2)} + \lambda\sigma\gamma\tau]$ , then  $\theta_n \rightarrow \theta$  as

$n \rightarrow \infty$ . It follows from (2.3.6) that  $\theta < 1$ . Hence  $\theta_n < 1$  for  $n$  sufficiently large. Since  $\lim_{n \rightarrow +\infty} \|J_\lambda^{\phi_n}(z) - J_\lambda^{\phi_{n-1}}(z)\| = 0$ , it follows from (2.3.16) that  $\{x_n\}$  is a Cauchy sequence in  $H$ . Since  $H$  is complete we may suppose that  $x_n \rightarrow x \in H$ .

Now we prove that  $u_n \rightarrow u \in M(x)$ ,  $v_n \rightarrow v \in S(x)$  and  $w_n \rightarrow w \in T(x)$ . In fact, it follows from Algorithm 2.3.1 that

$$\|u_n - u_{n-1}\| \leq (1 + 1/n) \gamma \|x_n - x_{n-1}\|,$$

$$\|v_n - v_{n-1}\| \leq (1 + 1/n) h \|x_n - x_{n-1}\|,$$

$$\|w_n - w_{n-1}\| \leq (1 + 1/n) d \|x_n - x_{n-1}\|,$$

which implies that  $\{u_n\}$ ,  $\{v_n\}$  and  $\{w_n\}$  are also Cauchy sequences in  $H$ . Let  $u_n \rightarrow u$ ,  $v_n \rightarrow v$ ,  $w_n \rightarrow w$  as  $n \rightarrow \infty$ . We have

$$\begin{aligned} d(v, S(x)) &= \inf\{\|v - y\| : y \in S(x)\} \\ &\leq \|v - v_n\| + d(v_n, S(x)) \\ &\leq \|v - v_n\| + \mathcal{H}(S(x_n), S(x)) \\ &\leq \|v - v_n\| + h \|x_n - x\| \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Hence  $v \in S(x)$ . Similarly we can prove that  $u \in M(x)$ ,  $w \in T(x)$ . This completes the proof.  $\square$

## 2.4 Iterative Algorithms and Convergence Results for Random Completely Generalized Nonlinear Variational-like Inclusions

The variational inequalities and quasi-variational inequalities for the random operators are called *random variational inequalities* and *random quasi-variational inequalities*; See for example [17, 32, 35, 43, 44, 50, 52, 94, 95, 96, 103] and references therein. Tan et al [94] studied random variational inequalities with applications to random minimization and nonlinear boundary problems, while Tarafdar and Yuan [96] gave

the applications of random variational inequalities to random best approximation and fixed point theorems. In [52] and [103], random quasi-variational inequalities are studied with applications to random generalized games.

In this section, we consider the completely generalized nonlinear variational-like inclusions for noncompact valued random mappings and suggest new iterative algorithm to compute the approximate solutions of our problem. We prove the existence of a random solution of our random completely generalized nonlinear variational-like inclusion and we study the convergence of random iterative sequences generated by the suggested algorithm. Several special cases are also considered.

Throughout this section, let  $(\Omega, \Sigma)$  be a measurable space, where  $\Omega$  is a set and  $\Sigma$  is a  $\sigma$ -algebra of subsets of  $\Omega$ . Let  $H$  be a real separable Hilbert space whose norm and inner product are denoted by  $\|\cdot\|$  and  $\langle \cdot, \cdot \rangle$ , respectively. We denote by  $\mathcal{B}(H)$  the class of Borel  $\sigma$ -field in  $H$ .

**Definition 2.4.1.** A mapping  $x : \Omega \rightarrow H$  is said to be *measurable* if for any  $B \in \mathcal{B}(H)$ ,  $\{t \in \Omega : x(t) \in B\} \in \Sigma$ .

**Definition 2.4.2.** A mapping  $f : \Omega \times H \rightarrow H$  is called a *random operator* if for any  $x \in H$ ,  $f(t, x) = x(t)$  is measurable. A random operator  $f$  is said to be *continuous* if for any  $t \in \Omega$ , the mapping  $f(t, \cdot) : H \rightarrow H$  is continuous.

**Definition 2.4.3.** A multivalued map  $T : \Omega \rightarrow 2^H$  is said to be *measurable* if for any  $B \in \mathcal{B}(H)$ ,  $T^{-1}(B) = \{t \in \Omega : T(t) \cap B \neq \emptyset\} \in \Sigma$ .

**Definition 2.4.4.** A mapping  $u : \Omega \rightarrow H$  is called a *measurable selection* of a multivalued measurable map  $T : \Omega \rightarrow 2^H$  if  $u$  is measurable and for any  $t \in \Omega$ ,  $u(t) \in T(t)$ .

**Definition 2.4.5.** A map  $T : \Omega \rightarrow 2^H$  is called a *random multivalued map* if for any  $x \in H$ ,  $T(\cdot, x)$  is measurable. A random multivalued map  $T : \Omega \times H \rightarrow CB(H)$  is said to be  *$\mathcal{H}$ -continuous* if for any  $t \in \Omega$ ,  $T(t, \cdot)$  is continuous in the Hausdorff metric.

Given random multivalued maps  $M, S, T : \Omega \times H \rightarrow 2^H$ , random operators  $f, g, p : \Omega \times H \rightarrow H$  with  $Im(g) \cap dom \partial\phi \neq \emptyset$  and the random map  $\eta : \Omega \times H \times H \rightarrow H$ ,



we consider the following *random completely generalized nonlinear variational-like inclusion problem*:

$$(CGNRVLIP) \quad \left\{ \begin{array}{l} \text{Find measurable mappings } x, u, v, w : \Omega \rightarrow H \text{ such that} \\ \forall t \in \Omega \text{ and } y(t) \in H \\ x(t) \in H, u(t) \in M(t, x(t)), v(t) \in S(t, x(t)), \\ w(t) \in T(t, x(t)), g(t, u(t)) \cap \text{dom } \partial\phi \neq \emptyset \text{ and} \\ \langle p(t, u(t)) - (f(t, v(t)) - g(t, w(t))), \eta(t, y(t), x(t)) \rangle \geq \phi(x(t)) - \phi(y(t)), \end{array} \right.$$

where  $\partial\phi$  is the subdifferential of a proper, convex and lower semicontinuous function  $\phi : H \rightarrow \mathbb{R} \cup \{+\infty\}$ . The set of measurable mappings  $(x, u, v, w)$  is called a *random solution* of (CGNRVLIP).

If  $p \equiv 0$ ,  $f$ ,  $g$  and  $M$  are identity maps, and  $S$  and  $T$  are single valued mappings, then (CGNRVLIP) reduces to the problem of finding a measurable mapping  $x : \Omega \rightarrow H$  such that  $\forall t \in \Omega$  and  $\forall y(t) \in H$ ,

$$\langle T(t, x(t)) - S(t, x(t)), \eta(t, y(t), x(t)) \rangle \geq 0. \quad (2.4.1)$$

It is considered and studied by Ding [35] in the setting of Banach spaces. He applied a random minimax inequality due to Tarafdar and Yuan [96] to prove existence and uniqueness theorems for the random solution of (2.4.1). By using the auxiliary principle technique, he suggested and analyzed an algorithm to compute the approximate random solutions of (2.4.1) in the setting of Banach spaces. He also discussed the convergence criteria.

We now give the following lemmas.

**Lemma 2.4.1.** [17] *Let  $T : \Omega \times H \rightarrow CB(H)$  be a  $\mathcal{H}$ -continuous random multivalued map. Then for any measurable mapping  $w : \Omega \rightarrow H$ , the multivalued map  $T(., w(.)) : \Omega \rightarrow CB(H)$  is measurable.*

**Lemma 2.4.2.** [17] *Let  $S, T : \Omega \rightarrow CB(H)$  be two measurable multivalued maps,  $\epsilon > 0$  be constant and  $v : \Omega \rightarrow H$  be a measurable selection of  $S$ . Then there exists a measurable selection  $w : \Omega \rightarrow H$  of  $T$  such that  $\forall t \in \Omega$ ,*

$$\|v(t) - w(t)\| \leq (1 + \epsilon)\mathcal{H}(S(t), T(t)).$$

**Lemma 2.4.3.** *The set of measurable mappings  $x, u, v, w : \Omega \rightarrow H$  is a random solution of (CGNRVLIP) if and only if  $\forall t \in \Omega, x(t) \in H, u(t) \in M(t, x(t)), v(t) \in S(t, x(t)), w(t) \in T(t, x(t))$  satisfy the following relation:*

$$x(t) = J_{\lambda(t)}^{\phi}[x(t) - \lambda(t)(p(t, u(t)) - (f(t, v(t)) - g(t, w(t))))], \quad (2.4.2)$$

where  $\lambda : \Omega \rightarrow (0, \infty)$  is a measurable mapping and  $J_{\lambda(t)}^{\phi} = [I + \lambda(t)\partial\phi]^{-1}$  is so called proximal map on  $H$  and  $I$  stands for the identity operator on  $H$ .

*Proof.* From the definition of  $J_{\lambda(t)}^{\phi}$ , it follows that

$$x(t) - \lambda(t)[p(t, u(t)) - (f(t, v(t)) - g(t, w(t)))] \in x(t) + \lambda(t)\partial_{\eta}\phi x(t)$$

and hence

$$[f(t, v(t)) - g(t, w(t))] - p(t, u(t)) \in \lambda(t)\partial_{\eta}\phi x(t).$$

By using the definition of  $\eta$ -subdifferential, we have

$$\langle (f(t, v(t)) - g(t, w(t))) - p(t, u(t)), \eta(t, y(t), x(t)) \rangle \leq \phi(y(t)) - \phi(x(t)) \quad \forall y(t) \in H, t \in \Omega.$$

Thus  $(x, u, v, w)$  is a random solution of (CGNRVLIP).  $\square$

We need the following definitions and results.

**Definition 2.4.6.** A random mapping  $\eta : \Omega \times H \times H \rightarrow H$  is called:

(i) *monotone* if

$$\langle x(t) - y(t), \eta(t, x(t), y(t)) \rangle \geq 0, \quad \forall x(t), y(t) \in H, t \in \Omega; \quad (2.4.3)$$

(ii) *strictly monotone* if the equality holds in (2.4.3) only when  $x(t) = y(t)$ ;

(iii) *strongly monotone* if there exists a measurable function  $q : \Omega \rightarrow (0, \infty)$  such that

$$\langle x(t) - y(t), \eta(t, x(t), y(t)) \rangle \geq q(t)\|x(t) - y(t)\|^2, \quad \forall x(t), y(t) \in H, t \in \Omega;$$

- (iv) *Lipschitz continuous* if there exists a measurable function  $s : \Omega \rightarrow (0, \infty)$  such that

$$\|\eta(t, x(t), y(t))\| \leq \delta \|x(t) - y(t)\|, \quad \forall x(t), y(t) \in H, t \in \Omega.$$

**Definition 2.4.7.** A random operator  $g : \Omega \times H \rightarrow H$  is said to be *Lipschitz continuous* if there exists a measurable function  $r : \Omega \rightarrow (0, \infty)$  such that

$$\|g(t, w_1(t)) - g(t, w_2(t))\| \leq r(t) \|w_1(t) - w_2(t)\|, \quad \forall w_1(t), w_2(t) \in H, t \in \Omega.$$

**Definition 2.4.8.** A random multivalued map  $S : \Omega \times H \rightarrow CB(H)$  is said to be  *$\mathcal{H}$ -Lipschitz continuous* if there exists a measurable function  $d : \Omega \rightarrow (0, \infty)$  such that

$$\mathcal{H}(S(t, x(t)), S(t, y(t))) \leq d(t) \|x(t) - y(t)\|, \quad \forall x(t), y(t) \in H, t \in \Omega.$$

**Definition 2.4.9.** Let  $f : H \rightarrow H$  be a random operator. A random multivalued map  $S : H \rightarrow 2^H$  is said to be:

- (i) *relaxed Lipschitz with respect to  $f$*  if there exists a measurable function  $k : \Omega \rightarrow (0, \infty)$  such that
- $$\langle f(t, u(t)) - f(t, v(t)), x - y \rangle \leq -k(t) \|x(t) - y(t)\|^2,$$
- $$\forall x(t), y(t) \in H, u(t) \in S(t, x(t)), v(t) \in S(t, y(t)), t \in \Omega;$$

- (ii) *relaxed monotone with respect to  $f$*  if there exists a measurable function  $c : \Omega \rightarrow (0, \infty)$  such that
- $$\langle f(t, u(t)) - f(t, v(t)), x(t) - y(t) \rangle \geq -c(t) \|x(t) - y(t)\|^2,$$
- $$\forall x(t), y(t) \in H, u(t) \in S(t, x(t)), v(t) \in S(t, y(t)), t \in \Omega.$$

**Definition 2.4.10.** Let  $\eta : \Omega \times H \times H \rightarrow H$  be a given random map. A random multivalued map  $Q : \Omega \times H \rightarrow 2^H$  is called  *$\eta$ -monotone* if  $\forall x(t), y(t) \in H$  and  $t \in \Omega$ ,

$$\langle u(t) - v(t), \eta(t, x(t), y(t)) \rangle \geq 0, \quad \forall u(t) \in Q(t, x(t)), v(t) \in Q(t, y(t)).$$

$Q$  is called *maximal  $\eta$ -monotone* if and only if it is  $\eta$ -monotone and there is no other  $\eta$ -monotone random multivalued map whose graph strictly contains the graph of  $Q$ .

**Assumption 2.4.1.** *The random operator  $\eta : \Omega \times H \times H \rightarrow H$  satisfies the condition*

$$\eta(t, y(t), x(t)) + \eta(t, x(t), y(t)) = 0, \quad \forall x(t), y(t) \in H, t \in \Omega.$$

*Remark 2.4.1.* If  $\eta : \Omega \times H \times H \rightarrow H$  satisfies Assumption 2.4.1 and  $\phi : H \rightarrow \mathbb{R} \cup \{+\infty\}$ , then it is easy to see that the random multivalued map  $\partial_\eta \phi : H \rightarrow 2^H$  is  $\eta$ -monotone.

The following result is the random version of Proposition 2.3.1 due to Lee, Ansari and Yao [63].

**Proposition 2.4.4.** *Let  $\eta : \Omega \times H \times H \rightarrow H$  be strictly monotone random map and  $Q : \Omega \times H \rightarrow 2^H$  an  $\eta$ -monotone random multivalued map. If, the range of  $(I + \lambda Q)$ ,  $R(I + \lambda Q) = H$ , for  $\lambda > 0$  and  $I$  is the identity operator, then  $Q$  is maximal  $\eta$ -monotone. Furthermore, the inverse operator  $(I + \lambda Q)^{-1}$  is single valued.*

Throughout this section, we will assume that  $\eta : \Omega \times H \times H \rightarrow H$  is strictly monotone and satisfies Assumption 2.4.1 and  $\phi : H \rightarrow \mathbb{R} \cup \{+\infty\}$  is a functional such that  $R(I + \lambda \partial_\eta \phi) = H$  for  $\lambda > 0$ .

From Proposition 2.4.4, we note that the mapping

$$J_\lambda^\phi(x(t)) = (I + \lambda \partial_\eta \phi)^{-1}(x(t)), \quad \forall x(t) \in H, t \in \Omega$$

is single valued.

To obtain the approximate solutions of (CGNRVLIP) we can apply a successive approximation method to the problem of solving

$$x(t) \in Q(t, x(t)) \tag{2.4.4}$$

for all  $t \in \Omega$ , where

$$Q(t, x(t)) = x(t) + J_{\lambda(t)}^\phi[x(t) - \lambda(t)((p(t, u(t)) - (f(t, S(t, v(t))) - g(t, T(t, w(t))))].$$

Based on (2.4.2) and (2.4.4), we propose the following algorithm to compute the approximate solutions of (CGNRVLIP).

**Algorithm 2.4.1.** Let  $M, S, T : \Omega \times H \rightarrow CB(H)$  be  $\mathcal{H}$ -continuous random multivalued maps and  $f, g, p : \Omega \times H \rightarrow H$  be continuous random operators. For any given measurable mapping  $x_0 : \Omega \rightarrow H$ , the multivalued mappings  $M(., x_0(.))$ ,  $S(., x_0(.))$ ,  $T(., x_0(.)) : \Omega \rightarrow CB(H)$  are measurable by Lemma 2.4.1. Hence there exist measurable selection  $u_0 : \Omega \rightarrow H$  of  $M(., x_0(.))$ , measurable selection  $v_0 : \Omega \rightarrow H$  of  $S(., x_0(.))$  and measurable selection  $w_0 : \Omega \rightarrow H$  of  $T(., x_0(.))$  by Himmelberg [42]. Let

$$x_1(t) = x_0(t) + J_{\lambda(t)}^{\phi_0}[x_0(t) - \lambda(t)((p(t, u_0(t)) - (f(t, v_0(t))) - g(t, w_0(t)))].$$

It is easy to see that  $x_1 : \Omega \rightarrow H$  is measurable. By Lemma 2.4.2, there exist measurable selections  $u_1 : \Omega \rightarrow H$  of  $M(., x_1(.))$ , measurable selection  $v_1 : \Omega \rightarrow H$  of  $S(., x_1(.))$  and measurable selection  $w_1 : \Omega \rightarrow H$  of  $T(., x_1(.))$  such that  $\forall t \in \Omega$ ,

$$\begin{aligned} \|u_0(t) - u_1(t)\| &\leq (1 + 1) \mathcal{H}(M(t, x_0(t)), M(t, x_1(t))), \\ \|v_0(t) - v_1(t)\| &\leq (1 + 1) \mathcal{H}(S(t, x_0(t)), S(t, x_1(t))), \\ \|w_0(t) - w_1(t)\| &\leq (1 + 1) \mathcal{H}(T(t, x_0(t)), T(t, x_1(t))). \end{aligned}$$

Let

$$x_2(t) = x_1(t) + J_{\lambda(t)}^{\phi_1}[x_1(t) - \lambda(t)((p(t, u_1(t)) - (f(t, v_1(t))) - g(t, w_1(t)))],$$

then  $x_2 : \Omega \rightarrow H$  is measurable.

By induction, we can obtain sequences  $\{x_n(t)\}, \{u_n(t)\}, \{v_n(t)\}$  and  $\{w_n(t)\}$  as follows:

$$x_{n+1}(t) = x_n(t) + J_{\lambda(t)}^{\phi_n}[x_n(t) - \lambda(t)((p(t, u_n(t)) - (f(t, v_n(t))) - g(t, w_n(t)))], \quad (2.4.5)$$

$$\begin{aligned} \|u_n(t) - u_{n+1}(t)\| &\leq (1 + (n + 1)^{-1}) \mathcal{H}(M(t, x_n(t)), M(t, x_{n+1}(t))), \\ \|v_n(t) - v_{n+1}(t)\| &\leq (1 + (n + 1)^{-1}) \mathcal{H}(S(t, x_n(t)), S(t, x_{n+1}(t))), \\ \|w_n(t) - w_{n+1}(t)\| &\leq (1 + (n + 1)^{-1}) \mathcal{H}(T(t, x_n(t)), T(t, x_{n+1}(t))), \end{aligned}$$

for any  $t \in \Omega$  and  $n = 0, 1, 2, \dots$ .

**Lemma 2.4.5.** *Let  $\eta : \Omega \times H \times H \rightarrow H$  be a strongly monotone and Lipschitz continuous random map with constants  $q(t) > 0$  and  $s(t) > 0$ , respectively, and satisfy Assumption (2.4.1). Then*

$$\|J_{\lambda(t)}^\phi x(t) - J_{\lambda(t)}^\phi y(t)\| \leq \tau(t) \|x(t) - y(t)\| \quad \forall x(t), y(t) \in H,$$

where  $\tau(t) = \frac{s(t)}{q(t)}$ .

**Theorem 2.4.6.** *Let  $\eta : \Omega \times H \times H \rightarrow H$  be strongly monotone and Lipschitz continuous random map with constants  $q(t) > 0$  and  $s(t) > 0$ , respectively, and satisfy Assumption (2.4.1). Let  $f, g, p : \Omega \times H \rightarrow H$  be Lipschitz continuous with corresponding constants  $\xi(t), r(t)$  and  $\sigma(t)$ , respectively. Let  $M, S, T : \Omega \times H \rightarrow CB(H)$  be  $\mathcal{H}$ -Lipschitz continuous with corresponding constants  $\gamma(t), h(t)$  and  $d(t)$ , respectively and  $S$  be the relaxed Lipschitz with respect to  $f$  with constant  $k(t)$  and  $T$  be relaxed monotone with respect to  $g$  with constant  $c(t)$ . For each  $n$ , let  $\phi_n : H \rightarrow \mathbb{R} \cup \{+\infty\}$  and  $\phi : H \rightarrow \mathbb{R} \cup \{+\infty\}$  be mappings such that  $R(I + \lambda(t)\partial_n \phi_n) = R(I + \lambda(t)\partial_n \phi) = H$  for  $\lambda(t) > 0$ . Assume that*

$$\lim_{n \rightarrow +\infty} \|J_{\lambda(t)}^{\phi_n} z(t) - J_{\lambda(t)}^{\phi} z(t)\| = 0, \quad \forall z(t) \in H$$

and if

$$\begin{aligned} & \left| \lambda(t) - \frac{k(t) - c(t)}{[\xi(t)h(t) + r(t)d(t)]^2 - \sigma^2(t)\gamma^2(t)} \right| \\ & < \frac{\sqrt{(c(t) - k(t))^2 - [(\xi(t)h(t) + r(t)d(t))^2 - \sigma^2(t)\gamma^2(t)]}}{[\xi(t)h(t) + r(t)d(t)]^2 - \sigma^2(t)\gamma^2(t)} \\ & c(t) - k(t) > \sqrt{[\xi(t)h(t) + r(t)d(t)]^2 - \sigma^2(t)\gamma^2(t)} \\ & \xi(t)h(t) + r(t)d(t) > \sigma(t)\gamma(t). \end{aligned} \tag{2.4.6}$$

Then there exists a set of elements  $x(t) \in H, u(t) \in M(t, x(t)), v(t) \in S(t, x(t))$  and  $w(t) \in T(t, x(t))$  which is a solution of (CGNRVLIP) and  $x_n(t) \rightarrow x(t), u_n(t) \rightarrow u(t), v_n(t) \rightarrow v(t), w_n(t) \rightarrow w(t)$  as  $n \rightarrow \infty$ , where  $\{x_n(t)\}, \{u_n(t)\}, \{v_n(t)\}$  and  $\{w_n(t)\}$  are the random sequences obtained by Algorithm (2.4.1).

*Proof.* From (2.4.5), we have

$$\|x_{n+1}(t) - x_n(t)\| = \|x_n(t) - x_{n-1}(t) + J_{\lambda(t)}^{\phi_n}(h(x_n(t))) - J_{\lambda(t)}^{\phi_{n-1}}(h(x_{n-1}(t)))\|, \quad (2.4.7)$$

where

$$h(x_n(t)) = x_n(t) - \lambda(t)[(p(t, u_n(t)) - (f(t, v_n(t)) - g(t, w_n(t)))).$$

By introducing the term  $J_{\lambda(t)}^{\phi_n}(h(x_{n-1}(t)))$ , we get

$$\begin{aligned} & \|J_{\lambda}^{\phi_n}(h(x_n(t))) - J_{\lambda}^{\phi_{n-1}}(h(x_{n-1}(t)))\| \\ & \leq \|J_{\lambda}^{\phi_n}(h(x_n(t))) - J_{\lambda}^{\phi_n}(h(x_{n-1}(t)))\| + \|J_{\lambda}^{\phi_n}(h(x_{n-1}(t))) - J_{\lambda}^{\phi_{n-1}}(h(x_{n-1}(t)))\|. \end{aligned} \quad (2.4.8)$$

By Lemma (2.4.5), we have

$$\begin{aligned} & \|J_{\lambda}^{\phi_n}(h(x_n(t))) - J_{\lambda}^{\phi_{n-1}}(h(x_{n-1}(t)))\| \\ & \leq \tau(t)\|h(x_n(t)) - h(x_{n-1}(t))\| + \|J_{\lambda}^{\phi_n}(h(x_{n-1}(t))) - J_{\lambda}^{\phi_{n-1}}(h(x_{n-1}(t)))\|, \end{aligned} \quad (2.4.9)$$

where  $\tau(t) = \frac{s(t)}{q(t)}$ , and

$$\begin{aligned} \|h(x_n(t)) - h(x_{n-1}(t))\| &= \|x_n(t) - \lambda(t)(p(t, u_n(t)) - (f(t, v_n(t)) \\ & \quad - g(t, w_n(t)))) - x_{n-1}(t) + \lambda(t)(p(t, u_{n-1}(t)) \\ & \quad - (f(t, v_{n-1}(t)) - g(t, w_{n-1}(t))))\| \\ &\leq \|x_n(t) - x_{n-1}(t) + \lambda(t)(f(t, v_n(t)) - f(t, v_{n-1}(t)) \\ & \quad - \lambda(t)(g(t, w_n(t)) - g(t, w_{n-1}(t)))\| \\ & \quad + \lambda(t)\tau(t)\|p(t, u_n(t)) - p(t, u_{n-1}(t))\|. \end{aligned} \quad (2.4.10)$$

From (2.4.7) - (2.4.10), we get

$$\begin{aligned} \|x_{n+1}(t) - x_n(t)\| &\leq \|x_n(t) - x_{n-1}(t)\| \\ & \quad + \tau(t)\|x_n(t) - x_{n-1}(t) + \lambda(t)(f(t, v_n(t)) \\ & \quad - f(t, v_{n-1}(t)) - \lambda(t)(g(t, w_n(t)) - g(t, w_{n-1}(t)))\| \\ & \quad + \lambda(t)\tau(t)\|p(t, u_n(t)) - p(t, u_{n-1}(t))\| \\ & \quad + \|J_{\lambda(t)}^{\phi_n}(h(x_{n-1}(t))) - J_{\lambda(t)}^{\phi_{n-1}}(h(x_{n-1}(t)))\|. \end{aligned} \quad (2.4.11)$$

Since  $M, S$  and  $T$  are  $\mathcal{H}$ - Lipschitz continuous, and  $f, g$  and  $p$  are Lipschitz continuous, we have

$$\begin{aligned} \|p(t, u_n(t)) - p(t, u_{n-1}(t))\| &\leq \sigma(t)\|u_n(t) - u_{n-1}(t)\| \\ &\leq \sigma(t)\gamma(t)(1 + 1/n)\|x_n(t) - x_{n-1}(t)\| \end{aligned} \quad (2.4.12)$$

$$\begin{aligned}
\|f(t, v_n(t)) - f(t, v_{n-1}(t))\| &\leq \xi(t)\|v_n(t) - v_{n-1}(t)\| \\
&\leq \xi(t)h(t)(1 + 1/n)\|x_n(t) - x_{n-1}(t)\|
\end{aligned} \tag{2.4.13}$$

$$\begin{aligned}
\|g(t, w_n(t)) - g(t, w_{n-1}(t))\| &\leq r(t)\|w_n(t) - w_{n-1}(t)\| \\
&\leq r(t)d(t)(1 + 1/n)\|x_n(t) - x_{n-1}(t)\|.
\end{aligned} \tag{2.4.14}$$

Further , since  $S$  is relaxed Lipschitz and  $T$  is relaxed monotone , we have

$$\begin{aligned}
&\|x_n(t) - x_{n-1}(t) + \lambda(t)(f(t, v_n(t)) - f(t, v_{n-1}(t)) - \lambda(t)(g(t, w_n(t)) - g(t, w_{n-1}(t)))\|^2 \\
&= \|x_n(t) - x_{n-1}(t)\|^2 + 2\lambda(t)\langle f(t, v_n(t)) - f(t, v_{n-1}(t)), x_n(t) \\
&\quad - x_{n-1}(t) \rangle - 2\lambda(t)\langle g(t, w_n(t)) - g(t, w_{n-1}(t)), x_n(t) - x_{n-1}(t) \rangle \\
&\quad + \lambda^2(t)\|f(t, v_n(t)) - f(t, v_{n-1}(t)) - (g(t, w_n(t)) - g(t, w_{n-1}(t)))\|^2 \\
&\leq [1 - 2\lambda(t)(k(t) - c(t)) + \lambda^2(t)(1 + 1/n)^2(\xi(t)h(t) \\
&\quad + r(t)d(t))^2]\|x_n(t) - x_{n-1}(t)\|^2.
\end{aligned} \tag{2.4.15}$$

From (2.4.11) - (2.4.15) , it follows that

$$\begin{aligned}
\|x_{n+1}(t) - x_n(t)\| &= \theta_n\|x_n(t) - x_{n-1}(t)\| \\
&\quad + \|J_{\lambda(t)}^{\phi_n}(h(x_{n-1}(t))) - J_{\lambda(t)}^{\phi_{n-1}}(h(x_{n-1}(t)))\|,
\end{aligned} \tag{2.4.16}$$

where

$$\begin{aligned}
\theta_n(t) &= 1 + \tau(t)\sqrt{(1 - 2\lambda(t))(k(t) - c(t)) + \lambda^2(t)(1 + 1/n)^2[\xi(t)h(t) + r(t)d(t)]^2} \\
&\quad + \lambda(t)\sigma(t)\gamma(t)\tau(t)(1 + 1/n).
\end{aligned}$$

Let,

$$\begin{aligned}
\theta(t) &= 1 + \tau(t)\sqrt{(1 - 2\lambda(t))(k(t) - c(t)) + \lambda^2(t)[\xi(t)h(t) + r(t)d(t)]^2} \\
&\quad + \lambda(t)\sigma(t)\gamma(t)\tau(t).
\end{aligned}$$

Then  $\theta_n(t) \rightarrow \theta(t)$  as  $n \rightarrow \infty$ . It follows from (2.4.6) that  $\theta(t) < 1$ .

Since  $\lim_{n \rightarrow +\infty} \|J_{\lambda(t)}^{\phi_n} z(t) - J_{\lambda(t)}^{\phi_{n-1}} z(t)\| = 0$ , It follows from (2.4.16) that  $\{x_n(t)\}$  is a Cauchy sequence in  $H$ . Since  $H$  is complete, then there exists a measurable map



$x : \Omega \rightarrow H$  such that  $x_n(t) \rightarrow x(t)$ , for all  $t \in \Omega$ . Now we prove that  $u_n(t) \rightarrow u(t) \in M(t, x(t))$ ,  $v_n(t) \rightarrow v(t) \in S(t, x(t))$  and  $w_n(t) \rightarrow w(t) \in T(t, x(t))$ . In fact, It follows from Algorithm (2.4.1) that

$$\|u_n(t) - u_{n-1}(t)\| \leq (1 + 1/n)\gamma(t)\|x_n(t) - x_{n-1}(t)\|,$$

$$\|v_n(t) - v_{n-1}(t)\| \leq (1 + 1/n)h(t)\|x_n(t) - x_{n-1}(t)\|,$$

$$\|w_n(t) - w_{n-1}(t)\| \leq (1 + 1/n)d(t)\|x_n(t) - x_{n-1}(t)\|,$$

which implies that  $\{u_n(t)\}$ ,  $\{v_n(t)\}$  and  $\{w_n(t)\}$  are also Cauchy sequences in  $H$ . Let  $u_n(t) \rightarrow u(t)$ ,  $v_n(t) \rightarrow v(t)$ ,  $w_n(t) \rightarrow w(t)$  as  $n \rightarrow \infty$ . We have

$$\begin{aligned} d(v(t), S(t, x(t))) &= \inf \{\|v(t) - y(t)\| : y \in S(t, x(t))\} \\ &\leq \|v(t) - v_n(t)\| + d(v_n(t), S(t, x(t))) \\ &\leq \|v(t) - v_n(t)\| + \mathcal{H}(S(t, x_n(t)), S(t, x(t))) \\ &\leq \|v(t) - v_n(t)\| + h\|x_n(t) - x(t)\| \rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

Hence  $v(t) \in S(t, x(t))$ . Similarly we can prove that  $u(t) \in M(t, x(t))$ ,  $w(t) \in T(t, x(t))$ . This complete the proof.  $\square$

## Chapter 3

# Generalized Nonlinear Quasi-variational Inclusions

This chapter deals with the iterative methods for computing the approximate solutions of generalized nonlinear quasi-variational inclusion problems. The existence and convergence of solutions obtained by proposed algorithms are also studied. We have also mentioned several special cases.

### 3.1 Introduction

There are many numerical methods to compute the approximate solutions of variational inequalities and their generalizes, for example, projection method and its variant forms, auxiliary principle method, Newton and descart framework etc. The applicability of projection method is limited to the variational inequalities without involving the nonlinear term  $\phi$  on the right hand side of the inequality. Such method cannot be used to suggest iterative algorithms for variational inequality problems of type (GSMVIP) and its generalized forms due to the presence of the nonlinear term  $\phi$ . In 1994, Hassouni and Moudafi [41] first used the resolvent operator instead of projection operator to suggest iterative algorithm for finding the approximate solutions of variational inequalities involving the nonlinear term  $\phi$  on the right hand side of the inequality. Adly [1] extended such approach for the more general variational inequality problems. The resolvent operators technique is extensively used by Noor,

Rassias, Huang, et al; See, for example [46, 75, 76, 79] and references therein. The resolvent operators technique is quite general and flexible.

Recently, Ding [31] considered a class of generalized quasi-variational inclusions. By using the properties of the resolvent operator, he established an existence result for solutions and suggested a new iterative algorithm and a perturbed proximal point algorithm for finding the approximate solutions. He also proved the convergence results for solutions obtained by the proposed algorithms.

The Mann [66] and Ishikawa [58] type perturbed iterative process for nonlinear equations is well documented in the literature. Huang [47] proposed Mann and Ishikawa types iterative algorithms for finding the approximate solutions of a more general problem known as generalized nonlinear implicit quasi-variational inclusions.

In this chapter, we consider generalized nonlinear quasi-variational inclusion problem. By using the resolvent operator technique, in the second section, we first convert our problem into a fixed point problem. Then we use this fixed point formulation to suggest an iterative algorithm for computing the approximate solutions of our problem. In the third section, we propose Ishikawa type perturbed iterative algorithm for finding the approximate solutions of our problem. The existence of solutions of our problem and the convergence results for approximate solutions obtained by the proposed algorithms are also studied. Several special cases of our problem are also mentioned.

Let  $H$  be a real Hilbert space whose norm and inner product are denoted by  $\|\cdot\|$  and  $\langle \cdot, \cdot \rangle$ , respectively. Given multivalued mappings  $A, S, T : H \rightarrow 2^H$  and single valued mappings  $g, m : H \rightarrow H$  and  $N : H \times H \rightarrow H$ , we consider the following problem which is called *generalized nonlinear quasi-variational inclusion problem*:

$$(GNQVIP) \quad \begin{cases} \text{Find } x \in H, u \in S(x), v \in T(x), \text{ and } w \in A(x) \text{ such that} \\ g(x) - m(w) \in \text{dom } \partial\phi \text{ and} \\ \langle N(u, v), y - g(x) \rangle \geq \phi(g(x) - m(w), x) - \phi(y, x), \quad \forall y \in H, \end{cases}$$

where  $\phi : H \times H \rightarrow \mathbb{R} \cup \{+\infty\}$  is a functional such that for each fixed  $y \in H$ ,  $\phi(\cdot, y) : H \rightarrow \mathbb{R} \cup \{+\infty\}$  is a proper, convex, lower semicontinuous functional on  $H$  and  $\partial\phi$  denotes the subdifferential of  $\phi$ .

SPECIAL CASES:

(i) If  $\phi(x, y) = \phi(x)$ ,  $\forall y \in H$ , then (GNQVIP) reduces to the following *generalized set-valued nonlinear quasi-variational inclusion problem* which is a special case of the problem considered by Shim et al [85] and appears to be a new one.

$$(GSVNQVIP) \quad \begin{cases} \text{Find } x \in H, u \in S(x), v \in T(x), \text{ and } w \in A(x) \text{ such that} \\ g(x) - m(w) \in \text{dom } \partial\phi \text{ and} \\ \langle N(u, v), y - g(x) \rangle \geq \phi(g(x) - m(w)) - \phi(y), \quad \forall y \in H. \end{cases}$$

(ii) If  $m \equiv 0$  and  $A$  is identity mapping, then (GSVNQVIP) reduces to the following *generalized set-valued mixed variational inequality problem* considered and studied by Noor et al [78]:

$$(GSMVIP) \quad \begin{cases} \text{Find } x \in H, u \in S(x), \text{ and } v \in T(x) \text{ such that} \\ g(x) \in \text{dom } \partial\phi \text{ and} \\ \langle N(u, v), y - g(x) \rangle \geq \phi(g(x)) - \phi(y), \quad \forall y \in H. \end{cases}$$

(iii) If  $N(u, v) = v - u$ ,  $m \equiv 0$  and  $A$  is identity mapping, then (GNQVIP) reduces to the following *generalized quasi-variational inclusion problem* considered and studied by Ding [31]:

$$(GQVIP) \quad \begin{cases} \text{Find } x \in H, u \in S(x), \text{ and } v \in T(x) \text{ such that} \\ g(x) \in \text{dom } \partial\phi \text{ and} \\ \langle u - v, y - g(x) \rangle \geq \phi(g(x), x) - \phi(y, x), \quad \forall y \in H. \end{cases}$$

For a suitable choice of the mappings  $A, S, T, N$ , and  $\phi$ , a number of other known variational inequalities and their generalizations can be obtained as special cases of our (GNQVIP).

### 3.2 Iterative Algorithms and Convergence Results

In this section, we first convert (GNQVIP) into a fixed point problem. Then by using this fixed point formulation, we suggest an iterative algorithm for computing

the approximate solutions of (GNQVIP). We also study the existence of solutions of (GNQVIP) and the convergence of approximate solutions obtained by suggested algorithm.

In order to prove our main result, we need the following concepts and results.

**Definition 3.2.1.** [98] Let  $S : H \rightarrow 2^H$  be a multivalued mapping. A mapping  $N(.,.) : H \times H \rightarrow H$  is said to be *relaxed Lipschitz with respect to  $S$*  in the first argument if there exists a constant  $k \geq 0$  such that

$$\langle N(u_1, .) - N(u_2, .), x - y \rangle \leq k\|x - y\|^2, \quad \forall u_1 \in S(x_1), u_2 \in S(x_2) \text{ and } x, y \in H.$$

**Definition 3.2.2.** [98] Let  $T : H \rightarrow 2^H$  be a multivalued mapping. A mapping  $N(.,.) : H \times H \rightarrow H$  is said to be *relaxed monotone with respect to  $T$*  in the second argument if there exists a constant  $c > 0$  such that

$$\langle N(., v_1) - N(., v_2), x - y \rangle \geq c\|x - y\|^2, \quad \forall v_1 \in T(x_1), v_2 \in T(x_2) \text{ and } x, y \in H.$$

The following lemma ensures that (GNQVIP) is equivalent to a fixed point problem.

**Lemma 3.2.1.** *The set of elements  $(x, u, v, w)$  is a solution of (GNQVIP) if and only if  $(x, u, v, w)$  satisfies the relation:*

$$g(x) = m(w) + J_{\eta}^{\partial\phi(.,x)}[g(x) - \eta N(u, v) - m(w)], \quad (3.2.1)$$

where  $\eta > 0$  is a constant and  $J_{\eta}^{\partial\phi(.,x)} = (I + \eta\partial\phi(.,x))^{-1}$  is the resolvent operator of  $\partial\phi(.,x)$  and  $I$  stands for the identity mapping on  $H$ .

The equation (3.2.1) can be written as

$$x = (1 - \lambda)x + \lambda[x - g(x) + m(w) + J_{\eta}^{\partial\phi(.,x)}[g(x) - \eta N(u, v) - m(w)]], \quad (3.2.2)$$

where  $0 < \lambda < 1$  and  $\eta > 0$  are both constants.

This fixed point formulation enables us to suggest the following algorithm.

**Algorithm 3.2.1.** Let  $g, m : H \rightarrow H$  be single valued mappings,  $N : H \times H \rightarrow H$  be a bifunction and  $A, S, T : H \rightarrow CB(H)$  be multivalued mappings. For given  $x_0 \in H$ , take  $u_0 \in S(x_0)$ ,  $v_0 \in T(x_0)$ ,  $w_0 \in A(x_0)$  and for  $\eta > 0$ , and let

$$x_1 = (1 - \lambda)x_0 + \lambda[x_0 - g(x_0) + m(w_0) + J_\eta^{\partial\phi(\cdot, x_0)}[g(x_0) - \eta N(u_0, v_0) - m(w_0)]],$$

where  $0 < \lambda < 1$  is a constant.

Since  $u_0 \in S(x_0) \in CB(H)$ ,  $v_0 \in T(x_0) \in CB(H)$ , and  $w_0 \in A(x_0) \in CB(H)$ , by Nadler [69], there exist  $u_1 \in S(x_1)$ ,  $v_1 \in T(x_1)$ , and  $w_1 \in A(x_1)$  such that

$$\|u_1 - u_0\| \leq (1 + 1)\mathcal{H}(S(x_1), S(x_0)),$$

$$\|v_1 - v_0\| \leq (1 + 1)\mathcal{H}(T(x_1), T(x_0)),$$

$$\|w_1 - w_0\| \leq (1 + 1)\mathcal{H}(A(x_1), A(x_0)).$$

Let

$$x_2 = (1 - \lambda)x_1 + \lambda[x_1 - g(x_1) + m(w_1) + J_\eta^{\partial\phi(\cdot, x_1)}[g(x_1) - \eta N(u_1, v_1) - m(w_1)]].$$

Since  $u_1 \in S(x_1) \in CB(H)$ ,  $v_1 \in T(x_1) \in CB(H)$ , and  $w_1 \in A(x_1) \in CB(H)$ , there exist,  $u_2 \in S(x_2)$ ,  $v_2 \in T(x_2)$ , and  $w_2 \in A(x_2)$  such that

$$\|u_1 - u_2\| \leq (1 + 2^{-1})\mathcal{H}(S(x_1), S(x_2)),$$

$$\|v_1 - v_2\| \leq (1 + 2^{-1})\mathcal{H}(T(x_1), T(x_2)),$$

$$\|w_1 - w_2\| \leq (1 + 2^{-1})\mathcal{H}(A(x_1), A(x_2)).$$

By induction, we can obtain sequences  $\{x_n\}$ ,  $\{u_n\}$ ,  $\{v_n\}$ , and  $\{w_n\}$  as

$$x_{n+1} = (1 - \lambda)x_n + \lambda[x_n - g(x_n) + m(w_n) + J_\eta^{\partial\phi(\cdot, x_n)}[g(x_n) - \eta N(u_n, v_n) - m(w_n)]], \quad (3.2.3)$$

$$u_n \in S(x_n), \quad \|u_n - u_{n+1}\| \leq (1 + (1 + n)^{-1})\mathcal{H}(S(x_n), S(x_{n+1})),$$

$$v_n \in T(x_n), \quad \|v_n - v_{n+1}\| \leq (1 + (1 + n)^{-1})\mathcal{H}(T(x_n), T(x_{n+1})),$$

$$w_n \in A(x_n), \quad \|w_n - w_{n+1}\| \leq (1 + (1 + n)^{-1})\mathcal{H}(A(x_n), A(x_{n+1})),$$

$n = 0, 1, 2, \dots$

Now we study the convergence of iterative sequences generated by Algorithm 3.2.1. and prove that existence of solutions of (GNQVIP)

**Theorem 3.2.2.** *Let  $g : H \rightarrow H$  be a strongly monotone and Lipschitz continuous mapping with constants  $\alpha > 0$  and  $\beta > 0$ , respectively, and  $m : H \rightarrow H$  be a Lipschitz continuous mapping with constant  $\gamma > 0$ . Let  $A, S, T : H \rightarrow CB(H)$  be  $\mathcal{H}$ -Lipschitz continuous mappings with constants  $\sigma > 0$ ,  $\xi > 0$  and  $\rho > 0$ , respectively. Let the bifunction  $N : H \times H \rightarrow H$  be relaxed Lipschitz continuous with respect to  $S$  in first argument with constant  $k \leq 0$ , and relaxed monotone with respect to  $T$  in second argument with constant  $c > 0$ . Also, let the bifunction  $N(.,.)$  be a Lipschitz continuous in first and second argument with constants  $\delta > 0$  and  $\omega > 0$ , respectively. Let  $\phi : H \times H \rightarrow \mathbb{R} \cup \{+\infty\}$  be such that for each fixed  $y \in H$ ,  $\phi(., y)$  is a proper, convex, lower semicontinuous function on  $H$ . For each  $x, y, z \in H$  and  $\eta > 0$ , let*

$$\|J_{\eta}^{\partial\phi(.,x)}(w) - J_{\eta}^{\partial\phi(.,y)}(w)\| \leq \mu\|x - y\|$$

and if

$$\begin{aligned} \left| \eta - \frac{(k-c)}{(\delta\xi + \omega\rho)^2} \right| &< \frac{\sqrt{(k-c)^2 - q(2-q)(\delta\xi + \omega\rho)^2}}{(\delta\xi + \omega\rho)^2}, \\ (k-c) &> (\delta\xi + \omega\rho)\sqrt{q(2-q)}, \\ q &= 2\sqrt{1-2\alpha+\beta^2} + 2\gamma\sigma + \mu, \quad q < 1, \end{aligned} \tag{3.2.4}$$

then there exist  $x \in H$ ,  $u \in S(x)$ ,  $v \in T(x)$ , and  $w \in A(x)$  satisfying the (GNQVIP). Moreover,

$$x_n \rightarrow x, \quad u_n \rightarrow u, \quad v_n \rightarrow v, \quad w_n \rightarrow w \quad \text{as } n \rightarrow \infty,$$

where the sequences  $\{x_n\}$ ,  $\{u_n\}$ ,  $\{v_n\}$  and  $\{w_n\}$  are generated by Algorithm 3.2.1.

*Proof.* From Algorithm 3.2.1, we have

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|(1-\lambda)x_n - (1-\lambda)x_{n-1} + \lambda[x_n - x_{n-1} - (g(x_n) - g(x_{n-1})) \\ &\quad + m(w_n) - m(w_{n-1}) + J_{\eta}^{\partial\phi(.,x_n)}[g(x_n) - \eta N(u_n, v_n) - m(w_n)] \\ &\quad - J_{\eta}^{\partial\phi(.,x_{n-1})}[g(x_{n-1}) - \eta N(u_{n-1}, v_{n-1}) - m(w_{n-1})]]\| \end{aligned}$$

$$\begin{aligned}
&\leq (1 - \lambda)\|x_n - x_{n-1}\| + \lambda\|x_n - x_{n-1} - (g(x_n) - g(x_{n-1}))\| \\
&\quad + \lambda\|m(w_n) - m(w_{n-1})\| + \lambda\|J_\eta^{\partial\phi(\cdot, x_n)}[g(x_n) - \eta N(u_n, v_n) - m(w_n)] \\
&\quad - J_\eta^{\partial\phi(\cdot, x_n)}[g(x_{n-1}) - \eta N(u_{n-1}, v_{n-1}) - m(w_{n-1})]\| \\
&\quad + \lambda\|J_\eta^{\partial\phi(\cdot, x_n)}[g(x_{n-1}) - \eta N(u_{n-1}, v_{n-1}) - m(w_{n-1})] \\
&\quad - J_\eta^{\partial\phi(\cdot, x_{n-1})}[g(x_{n-1}) - \eta N(u_{n-1}, v_{n-1}) - m(w_{n-1})]\| \\
&\leq (1 - \lambda)\|x_n - x_{n-1}\| + 2\lambda\|x_n - x_{n-1} - (g(x_n) - g(x_{n-1}))\| \\
&\quad + 2\lambda\|m(w_n) - m(w_{n-1})\| + \mu\lambda\|x_n - x_{n-1}\| + \lambda\|x_n - x_{n-1} \\
&\quad - \eta(N(u_n, v_n) - N(u_{n-1}, v_n) + N(u_{n-1}, v_n) - N(u_{n-1}, v_{n-1}))\| \\
&\leq (1 - \lambda)\|x_n - x_{n-1}\| + 2\lambda\|x_n - x_{n-1} - (g(x_n) - g(x_{n-1}))\| \\
&\quad + 2\gamma\lambda\|w_n - w_{n-1}\| + \mu\lambda\|x_n - x_{n-1}\| \\
&\quad + \lambda\|x_n - x_{n-1} - \eta(N(u_n, v_n) - N(u_{n-1}, v_n) \\
&\quad + N(u_{n-1}, v_n) - N(u_{n-1}, v_{n-1}))\|. \tag{3.2.5}
\end{aligned}$$

By the Lipschitz continuity and strong monotonicity of  $g$ , we obtain

$$\|x_n - x_{n-1} - (g(x_n) - g(x_{n-1}))\|^2 \leq (1 - 2\alpha + \beta^2)\|x_n - x_{n-1}\|^2. \tag{3.2.6}$$

Since  $A$ ,  $S$ , and  $T$  are  $\mathcal{H}$ -Lipschitz continuous and  $N$  is Lipschitz continuous in both the arguments, we have

$$\begin{aligned}
\|N(u_n, v_n) - N(u_{n-1}, v_n)\| &\leq \delta\|u_n - u_{n-1}\| \\
&\leq \delta\xi(1 + n^{-1})\|x_n - x_{n-1}\|. \tag{3.2.7}
\end{aligned}$$

$$\begin{aligned}
\|N(u_{n-1}, v_n) - N(u_{n-1}, v_{n-1})\| &\leq \omega\|v_n - v_{n-1}\| \\
&\leq \omega\rho(1 + n^{-1})\|x_n - x_{n-1}\| \tag{3.2.8}
\end{aligned}$$

and

$$\|w_n - w_{n-1}\| \leq \sigma(1 + n^{-1})\|x_n - x_{n-1}\|. \tag{3.2.9}$$

Since  $N$  is relaxed Lipschitz continuous with respect to  $S$  in first argument and relaxed monotone with respect to  $T$  in second argument, we have



$$\begin{aligned}
& \|x_n - x_{n-1} - \eta(N(u_n, v_n) - N(u_{n-1}, v_n) + N(u_{n-1}, v_n) - N(u_{n-1}, v_{n-1}))\|^2 \\
& \leq \|x_n - x_{n-1}\|^2 - 2\eta\langle N(u_n, v_n) - N(u_{n-1}, v_n), x_n - x_{n-1} \rangle \\
& \quad - 2\eta\langle N(u_{n-1}, v_n) - N(u_{n-1}, v_{n-1}), x_n - x_{n-1} \rangle \\
& \quad + \eta^2\|N(u_n, v_n) - N(u_{n-1}, v_n) + N(u_{n-1}, v_n) - N(u_{n-1}, v_{n-1})\|^2 \\
& \leq \|x_n - x_{n-1}\|^2 - 2\eta k\|x_n - x_{n-1}\|^2 + 2\eta c\|x_n - x_{n-1}\|^2 \\
& \quad + \eta^2(\delta\xi + \omega\rho)^2(1 + n^{-1})^2\|x_n - x_{n-1}\|^2 \\
& \leq [1 - 2\eta(k - c) + \eta^2(\delta\xi + \omega\rho)^2(1 + n^{-1})^2]\|x_n - x_{n-1}\|^2.
\end{aligned} \tag{3.2.10}$$

From (3.2.4) - (3.2.10), it follows that

$$\|x_{n+1} - x_n\| \leq \theta_n \|x_n - x_{n-1}\|, \tag{3.2.11}$$

where

$$\theta_n = \lambda q_n + \lambda \sqrt{1 - 2\eta(k - c) + \eta^2(\delta\xi + \omega\rho)^2(1 + n^{-1})^2} + (1 - \lambda)$$

and

$$q_n = 2\sqrt{1 - 2\alpha + \beta^2} + 2\gamma\sigma(1 + n^{-1}) + \mu.$$

Letting

$$\theta = \lambda q + \lambda \sqrt{1 - 2\eta(k - c) + \eta^2(\delta\xi + \omega\rho)^2} + (1 - \lambda).$$

We know that  $\theta_n \rightarrow \theta$  as  $n \rightarrow \infty$ . It follows from (3.2.4) that  $\theta < 1$ . Hence  $\theta_n < 1$  for  $n$  sufficiently large. Therefore (3.2.11) implies that  $\{x_n\}$  is a Cauchy sequence in  $H$  and we can suppose that  $x_n \rightarrow x \in H$  as  $n \rightarrow \infty$ .

Now we prove that  $u_n \rightarrow u \in S(x)$ ,  $v_n \rightarrow v \in T(x)$ , and  $w_n \rightarrow w \in A(x)$ . In fact, it follows from Algorithm 3.2.1, that

$$\|u_n - u_{n-1}\| \leq \xi(1 + n^{-1})\|x_n - x_{n-1}\|,$$

$$\|v_n - v_{n-1}\| \leq \lambda(1 + n^{-1})\|x_n - x_{n-1}\|,$$

$$\|w_n - w_{n-1}\| \leq \sigma(1 + n^{-1})\|x_n - x_{n-1}\|,$$

that is,  $\{u_n\}$ ,  $\{v_n\}$  and  $\{w_n\}$  are also Cauchy sequences in  $H$ . Let  $u_n \rightarrow u$ ,  $v_n \rightarrow v$ ,  $w_n \rightarrow w$  as  $n \rightarrow \infty$ . Further, we have

$$\begin{aligned} d(u, S(x)) &= \inf\{\|u - y\| : y \in S(x)\} \\ &\leq \|u - u_n\| + d(u_n, S(x)) \\ &\leq \|u - u_n\| + \mathcal{H}(S(x_n), S(x)) \\ &\leq \|u - u_n\| + \xi\|x_n - x\| \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Hence  $u \in S(x)$ . Similarly  $v \in T(x)$ , and  $w \in A(x)$ . From (3.2.3), we have

$$g(x) = m(w) + J_\eta^{\partial\phi(\cdot, x)}[g(x) - \eta N(u, v) - m(w)].$$

Therefore, it follows from Lemma 3.2.1 that the set of elements  $(x, u, v, w)$  is a solution of that (GNQVIP).  $\square$

### 3.3 An Ishikawa Type Perturbed Iterative Algorithm and a Convergence Result

In this section, we establish the equivalence of the generalized nonlinear quasi-variational inclusion (GNQVIP) to a nonlinear equation. Then we suggest an Ishikawa type perturbed iterative algorithm for finding the approximate solutions of (GNQVIP).

For finding the approximate solutions of (GNQVIP), we can apply a successive approximation method to the problem of solving

$$x \in \psi(x) \tag{3.3.1}$$

where

$$\psi(x) = x - g(x) + m(w) + J_\eta^{\partial\phi(\cdot, x)}[g(x) - \eta N(u, v) - m(w)].$$

On the basis of Lemma 3.2.1, we suggest the following Ishikawa type perturbed iterative algorithm.

**Algorithm 3.3.1.** (*Ishikawa Type Perturbed Iterative Algorithm*). Let  $g, m : H \rightarrow H$  be single valued mappings,  $N : H \times H \rightarrow H$  be a bifunction and  $A, S, T : H \rightarrow CB(H)$  be multivalued mappings.

For given  $x_0 \in H$ , we take  $u_0 \in S(x_0)$ ,  $v_0 \in T(x_0)$ , and  $w_0 \in A(x_0)$ , the iterative sequences  $\{x_n\}$ ,  $\{u_n\}$ ,  $\{v_n\}$ , and  $\{w_n\}$  are defined by

$$x_{n+1} = (1 - \lambda_n)x_n + \lambda_n[y_n - g(y_n) + m(\bar{w}_n) + J_\eta^{\partial\phi(\cdot, x_n)}[g(y_n) - \eta N(\bar{u}_n, \bar{v}_n) - m(\bar{w}_n)]] + e_n,$$

$$y_n = (1 - t_n)x_n + t_n[x_n - g(x_n) + m(w_n) + J_\eta^{\partial\phi(\cdot, y_n)}[g(x_n) - \eta N(u_n, v_n) - m(w_n)]] + t_n r_n,$$

for  $n \geq 0$ , where  $e_n$  and  $r_n$  are the error terms which are taken into account for a possible inexact computation of the points  $x_n \in H$ ,  $u_n \in S(x_n)$ ,  $\bar{u}_n \in S(y_n)$ ,  $v_n \in T(x_n)$ ,  $\bar{v}_n \in T(y_n)$  and  $w_n \in A(x_n)$ ,  $\bar{w}_n \in A(y_n)$ ,  $\rho > 0$  is a constant and  $\{\lambda_n\}$  and  $\{t_n\}$  are the sequences in  $[0, 1]$  satisfy the following conditions :

$$(i) \lambda_0 = 1, \quad 0 \leq \lambda_n, \quad t_n \leq 1, \quad \forall n \geq 0,$$

$$(ii) \sum_{n=0}^{\infty} \lambda_n \text{ diverges and}$$

$$(iii) \sum_{i=0}^n \prod_{j=i+1}^n (1 - (1 - c)\lambda_j) \text{ converges, where } 0 \leq c \leq 1.$$

**Theorem 3.3.1.** Let  $g : H \rightarrow H$  be a strongly monotone and Lipschitz continuous mapping with constant  $\alpha > 0$  and  $\beta > 0$ , respectively and  $m : H \rightarrow H$  be a Lipschitz continuous mapping with constant  $\gamma > 0$ . Let  $A, S, T : H \rightarrow CB(H)$  be  $\mathcal{H}$  Lipschitz continuous mappings with constant  $\sigma > 0$ ,  $\xi > 0$  and  $\rho > 0$ , respectively. Let the bifunction  $N : H \times H \rightarrow H$  be relaxed Lipschitz continuous with respect to  $S$  in first argument with constant  $k \leq 0$  and relaxed monotone with respect to  $T$  in second argument with constant  $c > 0$ . Also, let the bifunction  $N(\cdot, \cdot)$  be Lipschitz continuous in first and second argument with constant  $\delta > 0$  and  $w > 0$ , respectively. Let  $\phi : H \times H \rightarrow \mathbb{R} \cup \{+\infty\}$  be such that for each fixed  $y \in H$ ,  $\phi(\cdot, y)$  is a proper, convex, lower semicontinuous function on  $H$ . For each  $x, y, z \in H$  and  $\eta > 0$ , let

$$\|J_\eta^{\partial\phi(\cdot, x)}(z) - J_\eta^{\partial\phi(\cdot, y)}(z)\| \leq \mu\|x - y\|,$$

and if

$$\begin{aligned} \left| \eta - \frac{(k-c)}{(\delta\xi + w\rho)^2} \right| &< \frac{\sqrt{(k-c)^2 - q(2-q)(\delta\xi + w\rho)^2}}{(\delta\xi + w\rho)^2} \\ (k-c) &> (\delta\xi + w\rho)\sqrt{q(2-q)} \\ q &= 2\sqrt{1-2\alpha + \beta^2} + 2\gamma\sigma + \mu, \quad q < 1 \end{aligned} \quad (3.3.2)$$

then  $(x, u, v, w)$  is a solution of (GNQVIP). Moreover if

$$\lim_{n \rightarrow \infty} \|J_{\eta}^{\partial\phi(\cdot, x_n)}(y) - J_{\eta}^{\partial\phi(\cdot, x)}(y)\| = 0, \quad \forall y \in H,$$

and  $\{x_n\}, \{u_n\}, \{v_n\}$ , and  $\{w_n\}$  are defined by ITPIA with conditions

$$\begin{aligned} (a) \quad &\lim_{n \rightarrow \infty} \|e_n\| = 0 = \lim_{n \rightarrow \infty} \|r_n\| \quad \text{and} \\ (b) \quad &\sum_{i=0}^n \prod_{j=i+1}^n (1 - \lambda_j(1-c)) \text{ converges, } 0 \leq c < 1. \end{aligned}$$

Then  $\{x_n\}, \{u_n\}, \{v_n\}$ , and  $\{w_n\}$  are strongly converge to  $x, u, v$  and  $w$ , respectively.

*Proof.* First we prove that (GNQVIP) has a solution  $(x, u, v, w)$ . By Lemma (3.2.1) it is enough to show that mapping  $\psi : H \rightarrow 2^H$  defined by (3.3.1) has a fixed point  $x$ . For any  $x, y \in H, a \in \psi(x)$  and  $b \in \psi(y)$ , there exist  $u_1 \in S(x), u_2 \in S(y), v_1 \in T(x), v_2 \in T(y), w_1 \in A(x)$ , and  $w_2 \in A(y)$  such that

$$a = x - g(x) + m(w_1) + J_{\eta}^{\partial\phi(\cdot, x)}[g(x) - \eta N(u_1, v_1) - m(w_1)]$$

and

$$b = y - g(y) + m(w_2) + J_{\eta}^{\partial\phi(\cdot, y)}[g(y) - \eta N(u_2, v_2) - m(w_2)].$$

By Definition 1.2.6, we have

$$\begin{aligned} \|a - b\| &= \|x - g(x) + m(w_1) + J_{\eta}^{\partial\phi(\cdot, x)}[g(x) - \eta N(u_1, v_1) - m(w_1)] \\ &\quad - (y - g(y) + m(w_2) + J_{\eta}^{\partial\phi(\cdot, y)}[g(y) - \eta N(u_2, v_2) - m(w_2)])\| \end{aligned}$$

$$\begin{aligned}
&\leq \|x - y - (g(x) - g(y))\| + \|m(w_1) - m(w_2)\| \\
&\quad + \|J_\eta^{\partial\phi(\cdot, x)}[g(x) - \eta N(u_1, v_1) - m(w_1)] \\
&\quad - J_\eta^{\partial\phi(\cdot, x)}[g(y) - \eta N(u_2, v_2) - m(w_2)]\| \\
&\quad + \|J_\eta^{\partial\phi(\cdot, x)}[g(y) - \eta N(u_2, v_2) - m(w_2)] \\
&\quad - J_\eta^{\partial\phi(\cdot, y)}[g(y) - \eta N(u_2, v_2) - m(w_2)]\| \\
&\leq 2\|x - y - g(x) - g(y)\| + 2\gamma\|w_1 - w_2\| + \mu\|x - y\| \\
&\quad + \|x - y - \eta(N(u_1, v_1) - N(u_2, v_1) + N(u_2, v_1) - N(u_2, v_2))\|. \tag{3.3.3}
\end{aligned}$$

By Lipschitz continuity and strong monotonicity of  $g$ , we obtain

$$\|x - y - g(x) - g(y)\|^2 \leq (1 - 2\alpha + \beta^2)\|x - y\|^2. \tag{3.3.4}$$

Since  $A, S, T$  are  $\mathcal{H}$ -Lipschitz continuous and  $N$  is Lipschitz continuous in both the arguments, we have

$$\|N(u_1, v_1) - N(u_2, v_1)\| \leq \delta\|u_1 - u_2\| \leq \delta\xi\|x - y\| \tag{3.3.5}$$

$$\|N(u_2, v_1) - N(u_2, v_2)\| \leq w\|v_1 - v_2\| \leq w\rho\|x - y\| \tag{3.3.6}$$

and

$$\|w_1 - w_2\| \leq \sigma\|x - y\|. \tag{3.3.7}$$

Since  $N$  is relaxed Lipschitz continuous with respect to  $S$  in the first argument and relaxed monotone with respect to  $T$  in the second argument and from (3.3.5), (3.3.6), we have

$$\begin{aligned}
&\|x - y - \eta(N(u_1, v_1) - N(u_2, v_1) + N(u_2, v_1) - N(u_2, v_2))\|^2 \\
&\leq \|x - y\|^2 - 2\eta\langle N(u_1, v_1) - N(u_2, v_1), x - y \rangle - 2\eta\langle N(u_2, v_1) - N(u_2, v_2), x - y \rangle \\
&\quad + \eta^2\|N(u_1, v_1) - N(u_2, v_1) + N(u_2, v_1) - N(u_2, v_2)\|^2 \\
&\leq \|x - y\|^2 - 2\eta k\|x - y\|^2 + 2\eta c\|x - y\|^2 + \eta^2(\delta\xi + w\rho)^2\|x - y\|^2 \\
&\leq [1 - 2\eta(k - c) + \eta^2(\delta\xi + w\rho)^2]\|x - y\|^2. \tag{3.3.8}
\end{aligned}$$

From (3.3.3) - (3.3.8), it follows that

$$\|a - b\| \leq \theta\|x - y\|, \tag{3.3.9}$$

where

$$\theta = q + \sqrt{1 - 2\eta(k - c) + \eta^2(\delta\xi + w\rho)^2}$$

and

$$q = 2\sqrt{1 - 2\alpha + \beta^2} + 2\gamma\sigma + \mu.$$

By condition (3.3.2), we have  $0 < \theta_n < 1$ . It follows from (3.3.1) that  $\psi$  has a fixed point  $x \in H$ . By Lemma (3.2.1), there exist  $x \in H, u \in S(x), v \in T(x)$ , and  $w \in A(x)$  such that the set  $(x, u, v, w)$  is a solution of (GNQVIP).

Next, we prove that the iterative sequences  $\{x_n\}, \{u_n\}, \{v_n\}$ , and  $\{w_n\}$  defined by ITPIA strongly converge to  $x, u, v$ , and  $w$ , respectively. Since (GNQVIP) has a solution  $(x, u, v, w)$ , then by Lemma (3.2.1), we have

$$x = x - g(x) + m(w) + J_\eta^{\partial\phi(\cdot, x)}[g(x) - \eta N(u, v) - m(w)].$$

By making use of the same argument used for obtaining (3.3.4) and (3.3.8), we get

$$\|x_n - x - g(x_n) - g(x)\| \leq \sqrt{1 - 2\alpha + \beta^2} \|x_n - x\|,$$

$$\|x_n - x - \eta N(u_1, v_1) - \eta N(u_2, v_2)\| \leq \sqrt{1 - 2\eta(k - c) + \eta^2(\delta\xi + w\rho)^2} \|x_n - x\|,$$

$$\|y_n - x - g(y_n) - g(x)\| \leq \sqrt{1 - 2\alpha + \beta^2} \|y_n - x\|,$$

$$\|y_n - x - \eta N(u_1, v_1) - \eta N(u_2, v_2)\| \leq \sqrt{1 - 2\eta(k - c) + \eta^2(\delta\xi + w\rho)^2} \|y_n - x\|.$$

By setting

$$h(x) = g(x) - \eta N(u, v) - m(w)$$

$$h(y_n) = g(y_n) - \eta N(\bar{u}_n, \bar{v}_n) - m(\bar{w}_n).$$

We have

$$\begin{aligned} \|x_{n+1} - x\| &= \|(1 - \lambda_n)x_n + \lambda_n[y_n - g(y_n) + m(\bar{w}_n) + J_\eta^{\partial\phi(\cdot, x_n)}(h(y_n))]\| \\ &\quad + e_n - (1 - \lambda_n)x + \lambda_n[x - g(x) + m(w) + J_\eta^{\partial\phi(\cdot, x)}(h(x))]\| \\ &\leq (1 - \lambda_n)\|x_n - x\| + \lambda_n\|y_n - x - (g(y_n) - g(x))\| \\ &\quad + \lambda_n\|m(\bar{w}_n) - m(w)\| \\ &\quad + \lambda_n\|J_\eta^{\partial\phi(\cdot, x_n)}(h(y_n)) - J_\eta^{\partial\phi(\cdot, x)}(h(x))\| + \|e_n\|. \end{aligned} \tag{3.3.10}$$

Now since  $J_\eta^{\partial\phi(\cdot, x_n)}$  is nonexpansive, we have

$$\begin{aligned}
& \|J_\eta^{\partial\phi(\cdot, x_n)}(h(y_n)) - J_\eta^{\partial\phi(\cdot, x)}(h(x))\| \\
& \leq \|h(y_n) - h(x)\| + \|J_\eta^{\partial\phi(\cdot, x_n)}(h(x)) - J_\eta^{\partial\phi(\cdot, x)}(h(x))\| \\
& \leq \|y_n - x - (g(y_n) - g(x))\| + \|y_n - x - \eta(N(\bar{u}_n, \bar{v}_n) - N(u, v))\| \\
& \quad + \|m(\bar{w}_n) - m(w)\| + \|J_\eta^{\partial\phi(\cdot, x_n)}(h(x)) - J_\eta^{\partial\phi(\cdot, x)}(h(x))\| \\
& \leq \sqrt{1 - 2\alpha + \beta^2}\|y_n - x\| + \|y_n - x - \eta(N(\bar{u}_n, \bar{v}_n) - N(u_{n-1}, v_n) \\
& \quad + N(u_{n-1}, v_n) - N(u, v))\| + \gamma\|\bar{w}_n - w\| + \|J_\eta^{\partial\phi(\cdot, x_n)}(h(x)) - J_\eta^{\partial\phi(\cdot, x)}(h(x))\| \\
& \leq \sqrt{1 - 2\alpha + \beta^2}\|y_n - x\| + \sqrt{1 - 2\eta(k - c) + \eta^2(\delta\xi + w\rho)^2}\|y_n - x\| \\
& \quad + \gamma\sigma\|y_n - x\| + \mu\|y_n - x\| + \|J_\eta^{\partial\phi(\cdot, x_n)}(h(x)) - J_\eta^{\partial\phi(\cdot, y_n)}(h(x))\|. \tag{3.3.11}
\end{aligned}$$

By combining (3.3.10) and (3.3.11), we get

$$\begin{aligned}
\|x_{n+1} - x\| & \leq (1 - \lambda_n)\|x_n - x\| + \lambda_n[2\sqrt{1 - 2\alpha + \beta^2} \\
& \quad + \sqrt{1 - 2\eta(k - c) + \eta^2(\delta\xi + w\rho)^2} + 2\gamma\sigma + \mu]\|y_n - x\| \\
& \quad + \lambda_n\|J_\eta^{\partial\phi(\cdot, x_n)}(h(x)) - J_\eta^{\partial\phi(\cdot, y_n)}(h(x))\| + \|e_n\| \\
& = (1 - \lambda_n)\|x_n - x\| + \lambda_n\theta\|y_n - x\| + \lambda_n\varepsilon_n + \|e_n\|, \tag{3.3.12}
\end{aligned}$$

where

$$\theta = 2\sqrt{1 - 2\alpha + \beta^2} + \sqrt{1 - 2\eta(k - c) + \eta^2(\delta\xi + w\rho)^2} + 2\gamma\sigma + \mu$$

and

$$\varepsilon_n = \|J_\eta^{\partial\phi(\cdot, x_n)}(h(x)) - J_\eta^{\partial\phi(\cdot, y_n)}(h(x))\|.$$

Next,

$$\begin{aligned}
\|y_n - x\| & = \|(1 - t_n)x_n + t_n[x_n - g(x_n) + m(w_n) + J_\eta^{\partial\phi(\cdot, y_n)}(h(x_n))] + t_nr_n \\
& \quad - (1 - t_n)x + t_n[x - g(x) + m(w) + J_\eta^{\partial\phi(\cdot, x)}(h(x))]\| \\
& \leq (1 - t_n)\|x_n - x\| + t_n\|x_n - x - (g(x_n) - g(x))\| + t_n\|m(\bar{w}_n) - m(w)\| \\
& \quad + t_n\|J_\eta^{\partial\phi(\cdot, y_n)}(h(x_n)) - J_\eta^{\partial\phi(\cdot, x)}(h(x))\| + t_n\|r_n\|. \tag{3.3.13}
\end{aligned}$$

By making use of the same argument used for obtaining (3.3.11), we get

$$\begin{aligned}
& \|J_{\eta}^{\partial\phi(\cdot, y_n)}(h(x_n)) - J_{\eta}^{\partial\phi(\cdot, x)}(h(x))\| \\
& \leq \{ \sqrt{1 - 2\alpha + \beta^2} + \sqrt{1 - 2\eta(k - c) + \eta^2(\delta\xi + w\rho)^2} + \gamma\sigma + \mu \} \|x_n - x\| \\
& \quad + \varepsilon_n.
\end{aligned} \tag{3.3.14}$$

On combining (3.3.13) and (3.3.14), we get

$$\begin{aligned}
\|y_n - x\| & \leq (1 - t_n)\|x_n - x\| + t_n\theta\|x_n - x\| + t_n(\varepsilon_n + \|r_n\|) \\
& \leq (1 - t_n(1 - \theta))\|x_n - x\| + t_n(\varepsilon_n + \|r_n\|) \\
& \leq \|x_n - x\| + t_n(\varepsilon_n + \|r_n\|).
\end{aligned} \tag{3.3.15}$$

Since  $(1 - t_n(1 - \theta)) \leq 1$ , by combining (3.3.12) and (3.3.15), we get

$$\begin{aligned}
\|x_{n+1} - x\| & \leq (1 - \lambda_n)\|x_n - x\| + \lambda_n\theta\|x_n - x\| + \theta\lambda_nt_n(\varepsilon_n + \|r_n\|) + \lambda_n\varepsilon_n + \|e_n\| \\
& \leq (1 - \lambda_n(1 - \theta))\|x_n - x\| + \theta\lambda_nt_n(\varepsilon_n + \|r_n\|) + \lambda_n\varepsilon_n + \|e_n\| \\
& \leq \prod_{i=1}^n (1 - \lambda_i(1 - \theta))\|x_0 - x\| + \sum_{j=0}^n \lambda_j \prod_{i=j+1}^n (1 - \lambda_i(1 - \theta))\varepsilon_j \\
& \quad + \theta \sum_{j=0}^n \lambda_j t_j \prod_{i=j+1}^n (1 - \lambda_i(1 - \theta))(\varepsilon_j + \|r_j\|) \\
& \quad + \sum_{j=0}^n \prod_{i=j+1}^n (1 - \lambda_i(1 - \theta))\|e_j\|,
\end{aligned} \tag{3.3.16}$$

where

$$\prod_{i=j+1}^n (1 - \lambda_i(1 - \theta)) = 1, \quad \text{when } j = n.$$

Now, let  $B$  denote the lower triangular matrix with entries

$$b_{nj} = \lambda_j \sum_{i=j+1}^n (1 - \lambda_i(1 - \theta)).$$

Then  $B$  is multiplicative, see Rhodaes [84], so that

$$\lim_{n \rightarrow \infty} \sum_{j=0}^n \lambda_j \prod_{i=j+1}^n (1 - \alpha_i(1 - \theta))\varepsilon_j = 0,$$



$$\lim_{n \rightarrow \infty} \sum_{j=0}^n \lambda_j t_j \prod_{i=j+1}^n (1 - \lambda_i(1 - \theta))(\varepsilon_j + \|r_j\|) = 0.$$

Since

$$\lim_{n \rightarrow \infty} \|r_n\| = \lim_{n \rightarrow \infty} \varepsilon_n = 0.$$

Let  $D$  denote the lower triangular matrix with entries

$$d_{nj} = \prod_{i=j+1}^n (1 - \lambda_i(1 - \theta)).$$

The condition (b) implies that  $D$  is multiplicative and hence

$$\lim_{n \rightarrow \infty} \sum_{j=0}^n \prod_{i=j+1}^n (1 - \lambda_i(1 - \theta)) \|e_j\| = 0,$$

since

$$\lim_{n \rightarrow \infty} \|e_n\| = 0.$$

Also

$$\lim_{n \rightarrow \infty} \prod_{i=0}^n (1 - \lambda_i(1 - \theta)) = 0.$$

Since

$$\sum_{i=0}^n \lambda_i = \infty,$$

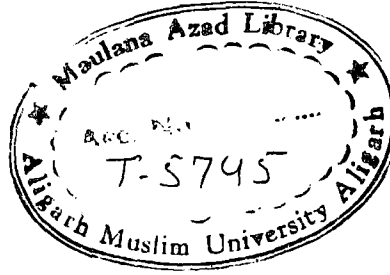
it follows that from inequality (3.3.16) that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x\| = 0,$$

that is, the sequence  $\{x_n\}$  strongly converges to  $x$  in  $H$ . The inequality (3.3.15) implies that the sequence  $\{y_n\}$  also converges to  $y$ . Since  $u_n \in S(x_n)$ ,  $u \in S(x)$ ,  $v_n \in T(x_n)$ ,  $v \in T(x)$  and  $w_n \in A(x_n)$ ,  $w \in A(x)$  and  $S, T, A$  are  $\mathcal{H}$  Lipschitz continuous, we have

$$\|u_n - u\| \leq \xi \|x_n - x\| \rightarrow 0 \text{ as } n \rightarrow \infty,$$

that is,  $\{u_n\}$  strongly converges to  $u$ . Similarly  $\{v_n\}$  and  $\{w_n\}$  strongly converge to  $v$  and  $w$ , respectively. This complete the proof.  $\square$



## Chapter 4

# Co-quasi-variational Inequalities

In this chapter, we consider two different classes of generalized co-quasi-variational inequalities in the setting of Banach spaces. By using the sunny nonexpansive retractions, we construct the projection iterative methods for finding the approximate solutions of our problems. Some existence and convergence results are also derived. In the last section, we consider the multivalued co-quasi-variational inequality problem for fuzzy mappings. Following the technique of the first two sections, we give an iterative algorithm and prove the convergence results for the approximate solutions obtained by our proposed algorithm. The existence result for a solution of this problem is also investigated.

### 4.1 Introduction

The projection iterative method is one of the most important and useful methods for finding the approximate solutions of fixed point problems, and variational and quasi-variational inequality problems; See for example [25, 38, 40, 56, 57, 60, 70, 71, 87, 88] and references therein. Most of the papers appeared in the literature on this topic, the metric projection operators, like in Hilbert spaces, are used. But it is impossible to use metric projection operator in the setting of Banach spaces because these operators are not nonexpansive. Recently, Takahashi and Kim [93] used sunny nonexpansive retraction to set up an iterative scheme for finding a fixed point of a nonexpansive

and nonself mappings in Banach spaces. Inspired by the work of Takahashi and Kim [93], Alber and Yao [3] used sunny nonexpansive retraction to construct the projection iterative method for finding the approximate solutions of a class of multivalued quasi-variational inequalities in Banach spaces. They gave the name *co-quasi-variational inequality* for quasi-variational inequality in Banach spaces and presented an iterative algorithm. They also proved several convergence results for approximate solutions obtained by their algorithm and in particular several existence results are obtained. Recently, Chang [19] also studied the existence of solutions and convergence of Mann and Ishikawa iterative processes for a class of variational inclusions with accretive type mappings in Banach spaces. The mathematical approach in [19] is quite different from the one used by Alber and Yao [3]. In this chapter, we mainly follow the approach of Alber and Yao [3].

## 4.2 Generalized Multivalued Mixed Co-quasi- variational Inequalities

In this section, we consider generalized multivalued mixed co-quasi-variational inequality problem in the setting of Banach spaces. By extending the terminology and method of Alber and Yao [3], we suggest and analyze an iterative algorithm to compute the approximate solution of our problem with noncompact valued mappings. We also prove convergence result for the approximate solutions obtained by our algorithm.

Let  $B$  be a real Banach space with its dual  $B^*$  and  $\langle x, f \rangle$  be a pairing between  $x \in B$  and  $f \in B^*$ . Given single valued mappings  $f, g, p, G : B \rightarrow B$  and multivalued mappings  $M, S, T, K : B \rightarrow 2^B$  such that  $\forall x \in B$ ,  $K(x)$  is nonempty, closed and convex subset of  $B$ , we consider the following *generalized multivalued mixed co-quasi-variational inequality problem*:

$$(GMMCQVIP) \quad \begin{cases} \text{Find } x \in B, u \in M(x), v \in S(x), \text{ and } w \in T(x) \\ \text{such that } G(x) \in K(x) \text{ and} \\ \langle p(u) - (f(v) - g(w)), J(z - G(x)) \rangle \geq 0, \quad \forall z \in K(x), \end{cases}$$

where  $J : B \rightarrow B^*$  is the normalized duality mapping. If  $B$  is a real Hilbert space, then (GMMCQVIP) reduces to the generalized multivalued mixed quasi-variational inequality problem which is more general and seems to be new one.

SPECIAL CASES:

(i) If  $p \equiv 0$ ,  $g$  and  $M$  are identity mappings and  $T$  is a single valued mapping, then (GMMCQVIP) reduces to the problem of finding  $x \in B$  and  $v \in S(x)$  such that  $G(x) \in K(x)$  and

$$\langle T(x) - f(v), J(z - G(x)) \rangle \geq 0, \quad \forall z \in K(x). \quad (4.2.1)$$

A problem similar to (4.2.1) is considered and studied by Alber and Yao [3]. They employed the sunny nonexpansive retraction method to formulate characterization of solutions of such problem. An iterative algorithm for finding the approximate solutions is suggested by them. They also derived some existence and convergence results.

(ii) If  $B$  is a Hilbert space,  $f, g$  and  $M$  are identity mappings,  $G(x) \equiv p(x)$  and  $K(x) = K$ , then (GMMCQVIP) reduces to the following *generalized variational inequality problem* considered and studied by Verma [98]:

$$(GVIP) \quad \begin{cases} \text{Find } x \in B, v \in S(x), \text{ and } w \in T(x) \text{ such that} \\ \langle p(x) - (v - w), z - p(x) \rangle \geq 0, \quad \forall z \in K. \end{cases}$$

It is clear that the generalized multivalued mixed co-quasi-variational inequality problem includes many kinds of quasi-variational inequalities, variational inequality and complementarity problems as special cases, such as in [40, 87, 98, 100] and references therein.

Now we first derive some characterizations for a solution of (GMMCQVIP).

**Lemma 4.2.1.** *Let  $B$  be a Banach space,  $f, g, p, G : B \rightarrow B$  be single valued mappings and  $M, S, T : B \rightarrow CB(B)$ , and  $K : B \rightarrow 2^B$  multivalued mappings such that  $\forall x \in B$ ,  $K(x)$  is a nonempty, closed and convex subset. Then the following statements are equivalent:*

(a) The set of elements  $(x, u, v, w)$  such that  $x \in B$ ,  $u \in M(x)$ ,  $v \in S(x)$  and  $w \in T(x)$ , is a solution of (GMMCQVIP).

(b)  $x \in B$ ,  $u \in M(x)$ ,  $v \in S(x)$ ,  $w \in T(x)$  and

$$G(x) = Q_{K(x)}[G(x) - \tau(p(u) - (f(v) - g(w)))] \text{ for any } \tau > 0.$$

For the proof of above lemma, we refer to [3] and references therein.

Combining Proposition 1.2.10 and Lemma 4.2.1, we obtain the following result on the characterization of solutions for (GMMCQVIP).

**Lemma 4.2.2.** *Let  $B$  be a real Banach space,  $X$  a nonempty, closed and convex subset of  $B$ . Let  $f, g, p, m, G : B \rightarrow B$  be single valued mappings, and  $M, S, T : B \rightarrow CB(B)$  and  $K : B \rightarrow 2^B$  be multivalued mappings such that  $\forall x \in B$ ,  $K(x) = m(x) + X$ . Then the set of elements  $(x, u, v, w)$  such that  $x \in B$ ,  $u \in M(x)$ ,  $v \in S(x)$  and  $w \in T(x)$  is a solution of (GMMCQVIP) if and only if*

$$x = x - G(x) + m(x) + Q_X[G(x) - \tau(p(u) - (f(v) - g(w))) - m(x)], \text{ for any } \tau > 0.$$

To compute the approximate solutions of (GMMCQVIP), we propose the following iterative algorithm.

**Algorithm 4.2.1.** *Let  $K(x) = m(x) + X$ , where  $X$  is a nonempty, closed and convex subset of  $B$  and  $\tau > 0$  be fixed. Let  $f, g, p, G : B \rightarrow B$  be single valued mappings and  $S, M, T : B \rightarrow CB(B)$  be multivalued mappings. For given  $u_0 \in M(x_0)$ ,  $v_0 \in S(x_0)$  and  $w_0 \in T(x_0)$ , we let*

$$x_1 = x_0 - G(x_0) + m(x_0) + Q_X[G(x_0) - \tau(p(u_0) - (f(v_0) - g(w_0))) - m(x_0)].$$

Since  $u_0 \in M(x_0) \in CB(B)$ ,  $v_0 \in S(x_0) \in CB(B)$ ,  $w_0 \in T(x_0) \in CB(B)$ , by Nadler [69], there exist  $u_1 \in M(x_1)$ ,  $v_1 \in S(x_1)$ ,  $w_1 \in T(x_1)$  such that

$$\|u_0 - u_1\| \leq (1 + 1)\mathcal{H}(M(x_0), M(x_1)),$$

$$\|v_0 - v_1\| \leq (1 + 1)\mathcal{H}(S(x_0), S(x_1)),$$

$$\|w_0 - w_1\| \leq (1 + 1)\mathcal{H}(T(x_0), T(x_1)).$$

Let

$$x_2 = x_1 - G(x_1) + m(x_1) + Q_X[G(x_1) - \tau(p(u_1) - (f(v_1) - g(w_1))) - m(x_1)].$$

By induction, we can obtain the sequences  $\{x_n\}$ ,  $\{u_n\}$ ,  $\{v_n\}$  and  $\{w_n\}$  as

$$x_{n+1} = x_n - G(x_n) + m(x_n) + Q_X[G(x_n) - \tau(p(u_n) - (f(v_n) - g(w_n))) - m(x_n)], \quad (4.2.2)$$

$$u_n \in M(x_n), \quad \|u_n - u_{n+1}\| \leq (1 + (n + 1)^{-1}) \mathcal{H}(M(x_n), M(x_{n+1})),$$

$$v_n \in S(x_n), \quad \|v_n - v_{n+1}\| \leq (1 + (n + 1)^{-1}) \mathcal{H}(S(x_n), S(x_{n+1})),$$

$$w_n \in T(x_n), \quad \|w_n - w_{n+1}\| \leq (1 + (n + 1)^{-1}) \mathcal{H}(T(x_n), T(x_{n+1})),$$

$$n = 0, 1, 2, 3, \dots$$

Now we prove that the approximate solutions of (GMMCQVIP) obtained by Algorithm (4.2.1) converge to the exact solution of (GMMCQVIP).

**Theorem 4.2.3.** *Let  $B$  be a real uniformly smooth Banach space with the module of smoothness  $\tau_B(t) \leq Dt^2$  for some  $D > 0$ . Let  $X$  be a nonempty, closed and convex subset of  $B$ ,  $f, g, p, G : B \rightarrow B$  be single valued mappings, and  $M, S, T : B \rightarrow CB(B)$  and  $K : B \rightarrow 2^B$  be multivalued mappings such that  $\forall x \in B$ ,  $K(x) = m(x) + X$ . Suppose that the following conditions are satisfied:*

- (i)  $f, g$ , and  $p$  are Lipschitz continuous with corresponding constants  $\xi$ ,  $r$  and  $\sigma$ , respectively.
- (ii)  $G$  is both strongly accretive with constant  $\gamma$  and Lipschitz continuous with constant  $\delta$ .
- (iii)  $M, S, T$  are  $\mathcal{H}$ -Lipschitz continuous with corresponding constants  $s, h$  and  $d$ , respectively.
- (iv)  $m$  is Lipschitz continuous with constant  $\theta$ .
- (v)  $0 < 2(1 - 2\gamma + 64D\delta^2)^{\frac{1}{2}} + 2\theta + \tau\sigma s + [1 + \tau(\xi h - rd)] < 1$ .

Then there exists a set of elements  $(x, u, v, w)$  such that  $u \in M(x)$ ,  $v \in S(x)$  and  $w \in T(x)$  which is a solution of (GMMCQVIP) and  $x_n \rightarrow x$ ,  $u_n \rightarrow u$ ,  $v_n \rightarrow v$ ,  $w_n \rightarrow w$  as  $n \rightarrow \infty$ , where  $\{x_n\}$ ,  $\{u_n\}$ ,  $\{v_n\}$  and  $\{w_n\}$  are the sequences obtained by Algorithm (4.2.1).

*Proof.* By the iterative scheme (4.2.2) and Proposition 1.2.10, we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|x_n - G(x_n) + m(x_n) + Q_X[G(x_n) - \tau(p(u_n) \\
&\quad - (f(v_n) - g(w_n))) - m(x_n)] - (x_{n-1} - G(x_{n-1}) + m(x_{n-1}) \\
&\quad - Q_X[G(x_{n-1}) - \tau(p(u_{n-1}) - (f(v_{n-1}) - g(w_{n-1}))) - m(x_{n-1}))]\| \\
&\leq \|x_n - x_{n-1} - (G(x_n) - G(x_{n-1}))\| + 2\|m(x_n) - m(x_{n-1})\| \\
&\quad + \|x_n - x_{n-1} - (G(x_n) - G(x_{n-1}))\| + \tau\|p(u_n) - p(u_{n-1})\| \\
&\quad + \|x_n - x_{n-1} + \tau(f(v_n) - f(v_{n-1})) - \tau(g(w_n) - g(w_{n-1}))\| \\
&= 2\|x_n - x_{n-1} - (G(x_n) - G(x_{n-1}))\| + 2\|m(x_n) - m(x_{n-1})\| \\
&\quad + \tau\|p(u_n) - p(u_{n-1})\| + \|x_n - x_{n-1} \\
&\quad + \tau(f(v_n) - f(v_{n-1})) - \tau(g(w_n) - g(w_{n-1}))\|. \tag{4.2.3}
\end{aligned}$$

By Proposition 1.2.8, we have (see, for example, the proof of Theorem 3 in [3]).

$$\|x_n - x_{n-1} - (G(x_n) - G(x_{n-1}))\|^2 \leq (1 - 2\gamma + 64D\delta^2)\|x_n - x_{n-1}\|^2. \tag{4.2.4}$$

It is clear that

$$\|m(x_n) - m(x_{n-1})\| \leq \theta\|x_n - x_{n-1}\|. \tag{4.2.5}$$

Since  $M, S$  and  $T$  are  $\mathcal{H}$ -Lipschitz continuous, and  $f, g$  and  $p$  are Lipschitz continuous, we have

$$\|p(u_n) - p(u_{n-1})\| \leq \sigma\|u_n - u_{n-1}\| \leq \sigma s \left(1 + \frac{1}{n}\right) \|x_n - x_{n-1}\|, \tag{4.2.6}$$

$$\|f(v_n) - f(v_{n-1})\| \leq \xi\|v_n - v_{n-1}\| \leq \xi h \left(1 + \frac{1}{n}\right) \|x_n - x_{n-1}\|, \tag{4.2.7}$$

$$\|g(w_n) - g(w_{n-1})\| \leq r\|w_n - w_{n-1}\| \leq rd \left(1 + \frac{1}{n}\right) \|x_n - x_{n-1}\|. \tag{4.2.8}$$

From (4.2.7) - (4.2.8), it follows that

$$\begin{aligned}
& \|x_n - x_{n-1} + \tau(f(v_n) - f(v_{n-1})) - \tau(g(w_n) - g(w_{n-1}))\| \\
& \leq \|x_n - x_{n-1}\| + \tau\|f(v_n) - f(v_{n-1})\| - 2\tau\|g(w_n) - g(w_{n-1})\| \\
& \leq \|x_n - x_{n-1}\| + \tau\xi h(1 + 1/n)\|x_n - x_{n-1}\| - \tau rd(1 + 1/n)\|x_n - x_{n-1}\| \\
& = [1 + \tau(1 + 1/n)(\xi h - rd)]\|x_n - x_{n-1}\|. \tag{4.2.9}
\end{aligned}$$

Combining (4.2.3) - (4.2.6) and (4.2.9), it follows that

$$\|x_{n+1} - x_n\| \leq t_n \|x_n - x_{n-1}\|,$$

where  $t_n = 2(1 - 2\gamma + 64D\delta^2)^{\frac{1}{2}} + 2\theta + \tau\sigma s(1 + \frac{1}{n}) + [1 + \tau(1 + 1/n)(\xi h - rd)]$ .

Let  $t = 2(1 - 2\gamma + 64D\delta^2)^{\frac{1}{2}} + 2\theta + \tau\sigma s + [1 + \tau(\xi h - rd)]$ . Then  $t_n \rightarrow t$  as  $n \rightarrow \infty$ . It follows from (v) that  $t < 1$ . Hence  $t_n < 1$  for  $n$  sufficiently large. Consequently  $\{x_n\}$  is a Cauchy sequence in  $B$ . Let  $x_n \rightarrow x \in B$ . Now we prove that  $u_n \rightarrow u \in M(x)$ ,  $v_n \rightarrow v \in S(x)$  and  $w_n \rightarrow w \in T(x)$ . In fact, it follows from Algorithm 4.2.2 that

$$\begin{aligned}
\|u_n - u_{n-1}\| & \leq \left(1 + \frac{1}{n}\right) s \|x_n - x_{n-1}\|, \\
\|v_n - v_{n-1}\| & \leq \left(1 + \frac{1}{n}\right) h \|x_n - x_{n-1}\|, \\
\|w_n - w_{n-1}\| & \leq \left(1 + \frac{1}{n}\right) d \|x_n - x_{n-1}\|,
\end{aligned}$$

which implies that  $\{u_n\}$ ,  $\{v_n\}$  and  $\{w_n\}$  are also Cauchy sequences in  $B$ . Let  $u_n \rightarrow u$ ,  $v_n \rightarrow v$ ,  $w_n \rightarrow w$  as  $n \rightarrow \infty$ . Since  $Q_X, G, f, g, p, M, S, T$  and  $m$  are continuous in  $B$ , we have

$$x = x - G(x) + m(x) + Q_X[G(x) - \tau(p(u) - (f(v) - g(w))) - m(x)].$$

Further, we have

$$\begin{aligned}
d(v, S(x)) &= \inf \{\|v - y\| : y \in S(x)\} \\
&\leq \|v - v_n\| + d(v_n, S(x)) \\
&\leq \|v - v_n\| + \mathcal{H}(S(x_n), S(x)) \\
&\leq \|v - v_n\| + h\|x_n - x\| \rightarrow 0 \text{ as } n \rightarrow \infty.
\end{aligned}$$

Hence  $v \in S(x)$ . Similarly we can prove that  $u \in M(x)$ ,  $w \in T(x)$ . The result then follows from Theorem 4.2.2.  $\square$



### 4.3 Completely Generalized Multivalued Co-quasi-variational Inequalities

In this section, we consider the completely generalized multivalued co-quasi-variational inequality problem in the setting of Banach spaces which is also considered and studied by Noor et al [77]. By using the sunny nonexpansive retraction operator, we give the characterization of solutions of our problem. This characterization is used to suggest and analyze an iterative algorithm for finding the approximate solutions of our problem. The existence and convergence results are also studied. Several special cases of our problem are also mentioned. Our approach is different from the one used by Noor et al [77].

Let  $B$  be a real Banach space with its dual  $B^*$  and  $\langle x, f \rangle$  be a pairing between  $x \in B$  and  $f \in B^*$ . Let  $N(.,.) : B \times B \rightarrow B$  and  $G : B \rightarrow B$  be nonlinear mappings and let  $T, A : B \rightarrow C(B)$  and  $K : B \rightarrow 2^B$  be multivalued mappings such that  $\forall x \in B$ ,  $K(x)$  is a nonempty, closed and convex set. We consider the following *completely generalized multivalued co-quasi-variational inequality problem* :

$$(CGMCQVIP) \quad \begin{cases} \text{Find } x \in B, u \in T(x), \text{ and } v \in A(x) \\ \text{such that } G(x) \in K(x) \text{ and} \\ \langle N(u, v), J(z - G(x)) \rangle \geq 0, \quad \forall z \in K(x), \end{cases}$$

where  $J : B \rightarrow B^*$  is the normalized duality operator. This problem is also considered by Noor et al [77].

SPECIAL CASES:

(i) If  $T$  is a single valued nonlinear operator and  $N(u, v) = T(x) + A(v)$ , then (CGMCQVIP) is equivalent to the following problem considered and studied by Alber and Yao [3]:

$$(GMCVIP) \quad \begin{cases} \text{Find } x \in B \text{ and } v \in A(x) \text{ such that} \\ G(x) \in K(x) \text{ and} \\ \langle T(x) + A(v), J(z - G(x)) \rangle \geq 0, \quad \forall z \in K(x). \end{cases}$$

It is called *generalized multi-valued co-variational inequality problem*. Alber and Yao [3] suggested an iterative algorithm for finding the approximate solutions of this problem. They also derived several convergence results for their algorithm and in particular several existence results for a solution of this problem.

(ii) When  $B$  is a Hilbert space,  $J$  reduces to the identity mapping. Consequently, (GMCVIP) reduces to the following problem:

$$(GMVVIP) \quad \begin{cases} \text{Find } x \in B \text{ and } v \in A(x) \text{ such that} \\ G(x) \in K(x) \text{ and} \\ \langle T(x) + A(v), z - G(x) \rangle \geq 0, \quad \forall z \in K(x). \end{cases}$$

It is called *generalized multi-valued variational inequality problem*, and it is introduced and studied by Jou and Yao [59].

It is clear that (CGMCQVIP) is more general and unifying one which includes many known problems considered and studied in the literature.

**Definition 4.3.1.** Let  $T, A : B \rightarrow C(B)$  be two multivalued mappings and  $N(.,.) : B \times B \rightarrow B$  be a nonlinear mapping. The mapping  $u \mapsto N(u, v)$  is said to be *strongly accretive with respect to  $T$*  if for any  $x_1, x_2 \in B$ , there exists a constant  $t > 0$  such that  $\forall u_1 \in T(x_1), u_2 \in T(x_2)$  and  $\forall v \in B$ ,

$$\langle N(u_1, v) - N(u_2, v), J(x_1 - x_2) \rangle \geq t \|x_1 - x_2\|^2.$$

**Definition 4.3.2.** The mapping  $N(.,.) : B \times B \rightarrow B$  is said to be *Lipschitz continuous with respect to first argument* if there exists a constant  $\beta > 0$  such that

$$\|N(u_1, .) - N(u_2, .)\| \leq \beta \|u_1 - u_2\|, \quad \forall u_1, u_2 \in B.$$

Now we first give some characterizations of solutions of (CGMCQVIP) by using sunny nonexpansive retraction operator. Then by using these characterizations, we suggest iterative algorithm for computing the approximate solutions of (CGMCQVIP).

**Lemma 4.3.1.** Let  $B$  be a real Banach space and  $N(.,.) : B \times B \rightarrow B$  be a nonlinear mapping. Let  $T, A : B \rightarrow C(B)$  and  $K : B \rightarrow 2^B$  be multivalued maps such that

$\forall x \in B$ ,  $K(x)$  is nonempty, closed and convex subset of  $B$ . Then the following statements are equivalent:

(a) The triplet  $(x, u, v)$ , where  $x \in B$ ,  $u \in T(x)$ , and  $v \in A(x)$ , is a solution of (CGMCQVIP).

(b)  $x \in B$ ,  $u \in T(x)$ ,  $v \in A(x)$  and  $G(x) = Q_{K(x)}[G(x) - \tau(N(u, v))]$  for any  $\tau > 0$ .

*Proof.* It is similar to the proof of Theorem 1 in [3] (see also the proof of Theorem 3.1 in [40] and Theorem 8.1 in [2]).  $\square$

By combining Proposition 1.2.10 and Lemma 4.3.1, we have the following result.

**Lemma 4.3.2.** *Let  $X$  be a nonempty, closed and convex subset of a real Banach space  $B$ . Let  $N(.,.) : B \times B \rightarrow B$  and  $g : B \rightarrow B$  be nonlinear mappings and  $T, A : B \rightarrow C(B)$  and  $K : B \rightarrow 2^B$  be multivalued maps such that  $\forall x \in B$ ,  $K(x) = m(x) + X$ . Then the triplet  $(x, u, v)$ , where  $x \in B$ ,  $u \in T(x)$ , and  $v \in A(x)$ , is a solution of (CGMCQVIP) if and only if*

$$x = x - G(x) + m(x) + Q_X[G(x) - \tau N(u, v) - m(x)], \quad \text{for any } \tau > 0.$$

We now construct the iterative algorithm for finding the approximate solutions of (CGMCQVIP).

**Algorithm 4.3.1.** *Let  $K(x) = m(x) + X$ , where  $X$  is a nonempty, closed and convex subset of a real Banach space  $B$  and  $\tau > 0$  be fixed. Given  $x_0 \in B$ , take any  $u_0 \in T(x_0)$ ,  $v_0 \in A(x_0)$  and let*

$$x_1 = x_0 - G(x_0) + m(x_0) + Q_X[G(x_0) - \tau N(u_0, v_0) - m(x_0)].$$

*Since  $T(x_0)$  and  $A(x_0)$  are nonempty and compact sets, there exist  $u_1 \in T(x_1)$  and  $v_1 \in A(x_1)$  such that*

$$\|u_0 - u_1\| \leq \mathcal{H}(T(x_0), T(x_1))$$

*and*

$$\|v_0 - v_1\| \leq \mathcal{H}(A(x_0), A(x_1)).$$

Let

$$x_2 = x_1 - G(x_1) + m(x_1) + Q_X[G(x_1) - \tau N(u_1, v_1) - m(x_1)].$$

By induction, we obtain sequences  $\{x_n\}$ ,  $\{u_n\}$  and  $\{v_n\}$  such that

$$u_n \in T(x_n), \quad \|u_n - u_{n+1}\| \leq \mathcal{H}(T(x_n), T(x_{n+1}))$$

$$v_n \in A(x_n), \quad \|v_n - v_{n+1}\| \leq \mathcal{H}(A(x_n), A(x_{n+1})),$$

and

$$x_{n+1} = x_n - G(x_n) + m(x_n) + Q_X[G(x_n) - \tau(N(u_n, v_n)) - m(x_n)], \quad (4.3.1)$$

$n = 0, 1, 2, \dots$

Next we prove that the approximate solutions of (CGMCQVIP) obtained by Algorithm 4.3.1 converge to the exact solution of (CGMCQVIP), and in particular, we prove the existence of a solution of (CGMCQVIP).

**Theorem 4.3.3.** *Let  $B$  be a real uniformly smooth Banach space with the module of smoothness  $\tau_B(t) \leq Ct^2$  for some  $C > 0$ . Let  $X$  be a nonempty, closed and convex subset of  $B$ ,  $N(.,.) : B \times B \rightarrow B$  and  $G, m : B \rightarrow B$  be nonlinear mappings and  $T, A : B \rightarrow C(B)$  and  $K : B \rightarrow 2^B$  be multivalued maps such that  $\forall x \in B$ ,  $K(x) = m(x) + X$ . Suppose that the following conditions are satisfied:*

- (i)  $N(.,.)$  is strongly accretive with respect to  $T$  and  $A$  with corresponding constants  $t > 0$  and  $s > 0$ , respectively.
- (ii)  $N(.,.)$  is Lipschitz continuous in both the arguments with corresponding constants  $\beta > 0$  and  $\alpha > 0$ , respectively.
- (iii)  $G$  is both strongly accretive with constant  $\gamma > 0$  and Lipschitz continuous with constant  $\delta > 0$ .
- (iv)  $m$  is Lipschitz continuous with constant  $\theta > 0$ .
- (v)  $T$  and  $A$  are  $\mathcal{H}$ -Lipschitz continuous with constant  $\xi > 0$  and  $\eta > 0$ , respectively.

$$(vi) \ 0 < 2(1 - 2\gamma + 64C\delta^2)^{\frac{1}{2}} + 2\theta + (1 - 2\tau(t + s) + 64C\tau^3(\alpha^2\eta^2 + \beta^2\xi^2))^{\frac{1}{2}} < 1.$$

Then there exist  $x \in B$ ,  $u \in T(x)$ , and  $v \in A(x)$  such that the triplet  $(x, u, v)$  is a solution of (CGMCQVIP) and the sequences  $\{x_n\}$ ,  $\{u_n\}$  and  $\{v_n\}$  generated by Algorithm 4.3.1 converge strongly to  $x$ ,  $u$  and  $v$ , respectively, that is,  $x_n \rightarrow x$ ,  $u_n \rightarrow u$  and  $v_n \rightarrow v$  as  $n \rightarrow \infty$ .

*Proof.* By the iterative scheme (4.3.1) and Proposition 1.2.10, we have

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|x_n - G(x_n) + m(x_n) + Q_X[G(x_n) - \tau N(u_n, v_n) - m(x_n)] \\ &\quad - x_{n-1} - G(x_{n-1}) + m(x_{n-1}) \\ &\quad - Q_X[G(x_{n-1}) - \tau N(u_{n-1}, v_{n-1}) - m(x_{n-1})]\| \\ &\leq \|x_n - x_{n-1} - (G(x_n) - G(x_{n-1}))\| + 2\|m(x_n) - m(x_{n-1})\| \\ &\quad + \|x_n - x_{n-1} - (G(x_n) - G(x_{n-1}))\| \\ &\quad + \|x_n - x_{n-1} - \tau(N(u_n, v_n) - N(u_{n-1}, v_{n-1}))\| \\ &= 2\|x_n - x_{n-1} - (G(x_n) - G(x_{n-1}))\| \\ &\quad + 2\|m(x_n) - m(x_{n-1})\| \\ &\quad + \|x_n - x_{n-1} - \tau(N(u_n, v_n) - N(u_{n-1}, v_{n-1}))\|. \end{aligned} \quad (4.3.2)$$

By Proposition 1.2.8, we have (see, for example, the proof of the Theorem 3 in [3])

$$\|x_n - x_{n-1} - (G(x_n) - G(x_{n-1}))\|^2 \leq (1 - 2\gamma + 64C\delta^2)\|x_n - x_{n-1}\|^2. \quad (4.3.3)$$

Since  $N(., .)$  is strongly accerative with respect to the mappings  $T$  and  $A$ ,  $N(., .)$  is Lipschitz continuous in both the arguments, by using Proposition 1.2.8 and Algorithm 4.3.1, we have

$$\begin{aligned} &\|x_n - x_{n-1} - \tau(N(u_n, v_n) - N(u_{n-1}, v_{n-1}))\|^2 \\ &\leq \|x_n - x_{n-1}\|^2 - 2\tau\langle N(u_n, v_n) - N(u_{n-1}, v_{n-1}), \\ &\quad J(x_n - x_{n-1}) - \tau(N(u_n, v_n) - N(u_{n-1}, v_{n-1})) \rangle \\ &= \|x_n - x_{n-1}\|^2 - 2\tau\langle N(u_n, v_n) - N(u_{n-1}, v_{n-1}), \\ &\quad J(x_n - x_{n-1}) \rangle - 2\tau\langle N(u_n, v_n) - N(u_{n-1}, v_{n-1}), J(x_n - x_{n-1}) \\ &\quad - \tau(N(u_n, v_n) - N(u_{n-1}, v_{n-1})) \rangle - J(x_n - x_{n-1}) \rangle \end{aligned}$$

$$\begin{aligned}
&= \|x_n - x_{n-1}\|^2 - 2\tau \langle N(u_n, v_n) - N(u_{n-1}, v_{n-1}), J(x_n - x_{n-1}) \rangle \\
&\quad + N(u_{n-1}, v_{n-1}) - N(u_{n-1}, v_{n-1}), J(x_n - x_{n-1}) \rangle \\
&\quad - 2\tau \langle N(u_n, v_n) - N(u_{n-1}, v_{n-1}), J(x_n - x_{n-1}) \rangle \\
&\quad - \tau \langle N(u_n, v_n) - N(u_{n-1}, v_{n-1}) \rangle - J(x_n - x_{n-1}) \rangle \\
&= \|x_n - x_{n-1}\|^2 - 2\tau \langle N(u_n, v_n) - N(u_{n-1}, v_{n-1}), J(x_n - x_{n-1}) \rangle \\
&\quad - 2\tau \langle N(u_{n-1}, v_{n-1}) - N(u_{n-1}, v_{n-1}), J(x_n - x_{n-1}) \rangle \\
&\quad - 2\tau \langle (N(u_n, v_n) - N(u_{n-1}, v_{n-1})), J(x_n - x_{n-1}) \rangle \\
&\quad - \tau \langle N(u_n, v_n) - N(u_{n-1}, v_{n-1}) \rangle - J(x_n - x_{n-1}) \rangle \\
&\leq \|x_n - x_{n-1}\|^2 - 2\tau t \|x_n - x_{n-1}\|^2 - 2\tau s \|x_n - x_{n-1}\|^2 \\
&\quad + 4d^2 \rho_B \tau \left( \frac{4\tau \|N(u_n, v_n) - N(u_{n-1}, v_{n-1})\|}{d} \right) \\
&= (1 - 2\tau(t + s)) \|x_n - x_{n-1}\|^2 + 4d^2 \rho_B \tau \left( \frac{4\tau}{d} \|N(u_n, v_n) \right. \\
&\quad \left. - N(u_n, v_{n-1}) + N(u_n, v_{n-1}) - N(u_{n-1}, v_{n-1})\| \right) \\
&\leq (1 - 2\tau(t + s)) \|x_n - x_{n-1}\|^2 \\
&\quad + 4d^2 \rho_B \tau \left( \frac{4\tau}{d} \|N(u_n, v_n) - N(u_n, v_{n-1})\| \right. \\
&\quad \left. + \|N(u_n, v_{n-1}) - N(u_{n-1}, v_{n-1})\| \right) \\
&\leq (1 - 2\tau(t + s)) \|x_n - x_{n-1}\|^2 + 64C\tau^3 (\|N(u_n, v_n) \\
&\quad - N(u_n, v_{n-1})\|^2 + \|N(u_n, v_{n-1}) - N(u_{n-1}, v_{n-1})\|^2) \\
&\leq (1 - 2\tau(t + s)) \|x_n - x_{n-1}\|^2 \\
&\quad + 64C\tau^3 (\alpha^2 \|v_n - v_{n-1}\|^2 + \beta^2 \|u_n - u_{n-1}\|^2) \\
&\leq (1 - 2\tau(t + s)) \|x_n - x_{n-1}\|^2 \\
&\quad + 64C\tau^3 (\alpha^2 \mathcal{H}^2(A(x_n), A(x_{n-1})) + \beta^2 \mathcal{H}^2(T(x_n), T(x_{n-1}))) \\
&\leq (1 - 2\tau(t + s)) \|x_n - x_{n-1}\|^2 \\
&\quad + 64C\tau^3 (\alpha^2 \eta^2 \|x_n - x_{n-1}\|^2 + \beta^2 \xi^2 \|x_n - x_{n-1}\|^2) \\
&= (1 - 2\tau(t + s) + 64C\tau^3 (\alpha^2 \eta^2 + \beta^2 \xi^2)) \|x_n - x_{n-1}\|^2. \tag{4.3.4}
\end{aligned}$$

It is clear from the Lipschitz continuity of  $m$  that

$$\|m(x_n) - m(x_{n-1})\| \leq \theta \|x_n - x_{n-1}\|. \tag{4.3.5}$$

From (4.3.3) - (4.3.5), we have the following inequality

$$\|x_{n+1} - x_n\| \leq k\|x_n - x_{n-1}\|,$$

where

$$k = 2(1 - 2\gamma + 64C\delta^2)^{\frac{1}{2}} + 2\theta + (1 - 2\tau(t + s) + 64C\tau^3(\alpha^2\eta^2 + \beta^2\xi^2))^{\frac{1}{2}}$$

and  $0 < k < 1$  by (vi). Consequently,  $\{x_n\}$  is a Cauchy sequence and thus converges to some  $x \in B$ . Now we prove that  $u_n \rightarrow u \in T(x)$  and  $v_n \rightarrow v \in A(x)$ . From Algorithm 4.3.1, we have

$$\|u_{n+1} - u_n\| \leq \mathcal{H}(T(x_{n+1}), T(x_n)) \leq \xi\|x_{n+1} - x_n\|$$

and

$$\|v_{n+1} - v_n\| \leq \mathcal{H}(A(x_{n+1}), A(x_n)) \leq \eta\|x_{n+1} - x_n\|$$

which imply that the sequences  $\{u_n\}$  and  $\{v_n\}$  are Cauchy in  $B$ . Let  $u_n \rightarrow u$  and  $v_n \rightarrow v$ . Since  $Q_X$ ,  $G$ ,  $T$ ,  $A$ ,  $N(.,.)$  and  $m$  are continuous, we have

$$x = x - G(x) + m(x) + Q_X[G(x) - \tau N(u, v) - m(x)].$$

It remains to show that  $u \in T(x)$  and  $v \in A(x)$ . In fact,

$$\begin{aligned} d(u, T(x)) &= \inf \{\|u - w\| : w \in T(x)\} \\ &\leq \|u - u_n\| + d(u_n, T(x)) \\ &\leq \|u - u_n\| + \mathcal{H}(T(x_n), T(x)) \\ &\leq \|u - u_n\| + \xi\|x_n - x\| \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Hence  $d(u, T(x)) = 0$  and therefore  $u \in T(x)$ . Similarly, we can prove that  $v \in A(x)$ . The result then follows from Lemma 4.3.2.  $\square$

## 4.4 Generalized Multivalued Co-quasi-variational Inequalities with Fuzzy Mappings

In an unpublished work, Ansari [4] introduced the concept of variational inequalities for fuzzy mappings, called *fuzzy variational inequalities*, in his Ph.D. thesis. Separately, Chang and Zhu [23] also studied a class of variational inequalities for fuzzy mappings. Several kinds of variational and quasi-variational inequalities for fuzzy mappings are considered and studied by Chang [18], Chang and Huang [22], Noor [73] and Lee et al [62]. Motivated and inspired by the work going in this field, in this section, we consider the multivalued co-quasi-variational inequality problem for fuzzy mappings in the setting of Banach spaces. Following the technique of previous sections, we give an iterative algorithm for computing the approximate solutions of our problem. The existence and convergence results are also studied.

Let  $B$  be a real Banach space and  $B^*$  be its topological dual space. Let  $\langle \cdot, \cdot \rangle$  be the dual pairing between  $B$  and  $B^*$ . Let  $\mathcal{F}(B)$  be a collection of all fuzzy sets over  $B$ . A mapping  $P : B \rightarrow \mathcal{F}(B)$  is said to be a *fuzzy mapping*. For each  $x \in B$ ,  $P(x)$  (denoted by  $P_x$  in the sequel) is a fuzzy set on  $B$  and  $P_x(y)$  is the membership function of  $y$  in  $P_x$ .

A fuzzy mapping  $P : B \rightarrow \mathcal{F}(B)$  is said to be *closed* if for each  $x \in B$ , the function  $y \mapsto P_x(y)$  is upper semicontinuous, that is, for any given net  $\{y_\alpha\} \subset B$  satisfying  $y_\alpha \rightarrow y_0 \in B$ ,  $\limsup_{\alpha} P_x(y_\alpha) \leq P_x(y_0)$ . For  $C \in \mathcal{F}(B)$  and  $\lambda \in [0, 1]$ , the set  $(C)_\lambda = \{x \in B : C(x) \geq \lambda\}$  is called a  $\lambda$ -cut set of  $C$ .

A closed fuzzy mapping  $A : B \rightarrow \mathcal{F}(B)$  is said to satisfy condition  $(*)$  : if there exists a function  $a : B \rightarrow [0, 1]$  such that for each  $x \in B$ ,  $(A_x)_{a(x)}$  is a nonempty and bounded subset of  $B$ .

It is clear that if  $A$  is a closed fuzzy mapping satisfying condition  $(*)$ , then for each  $x \in B$ , the set  $(A_x)_{a(x)} \in CB(B)$ .

In fact, let  $\{y_\alpha\}_{\alpha \in \Gamma} \subset (A_x)_{a(x)}$  be a net and  $y_\alpha \rightarrow y_0 \in B$ . Then  $(A_x)(y_\alpha) \geq a(x)$  for each  $\alpha \in \Gamma$ . Since  $A$  is closed, we have

$$A_x(y_0) \geq \limsup_{\alpha \in \Gamma} A_x(y_\alpha) \geq a(x).$$



This implies that  $y_0 \in (A_x)_{a(x)}$  and so  $(A_x)_{a(x)} \in CB(B)$ .

Let  $P, Q : B \rightarrow \mathcal{F}(B)$  be two closed fuzzy mappings satisfying condition (\*). Then there exist two functions  $a, b : B \rightarrow [0, 1]$  corresponding to  $P$  and  $Q$ , respectively, such that for each  $x \in B$ , we have  $(P_x)_{a(x)}, (Q_x)_{b(x)} \in CB(B)$ . Therefore, we can define two multivalued mappings  $\tilde{P}, \tilde{Q} : B \rightarrow CB(B)$  by

$$\tilde{P}(x) = (P_x)_{a(x)}, \quad \tilde{Q}(x) = (Q_x)_{b(x)}, \quad \forall x \in B.$$

In the sequel,  $\tilde{P}$  and  $\tilde{Q}$  are called the *multivalued mappings induced by the fuzzy mappings*  $P$  and  $Q$  respectively.

Let  $T, A, G : B \rightarrow B$  be single valued mappings,  $P, Q : B \rightarrow \mathcal{F}(B)$  be two fuzzy mappings. Let  $a, b : B \rightarrow [0, 1]$  be given functions. Let  $K : B \rightarrow 2^B$  such that  $\forall x \in B$ ,  $K(x)$  is a nonempty, closed and convex. We consider the following *multivalued fuzzy co-quasi-variational inequality problem*:

$$(MFCVQIP) \quad \begin{cases} \text{Find } x, u, \text{ and } v \in K \text{ such that} \\ P_x(u) \geq a(x), Q_x(v) \geq b(x), G(x) \in K(x) \text{ and} \\ \langle T(u) + A(v), J(z - G(x)) \rangle \geq 0 \quad \forall z \in K(x), \end{cases}$$

where  $J : B \rightarrow B^*$  is the normalized duality mapping.

**Lemma 4.4.1.** *Let  $B$  be a real Banach space,  $T, A, G : B \rightarrow B$  be single valued mappings,  $\tilde{P}, \tilde{Q} : B \rightarrow CB(B)$  and  $K : B \rightarrow 2^B$  be multivalued mappings such that  $\forall x \in B$ ,  $K(x)$ , is nonempty, closed and convex. Then the following statements are equivalent:*

- (a) *The set of elements  $x \in B, u \in \tilde{P}(x)$ , and  $v \in \tilde{Q}(x)$  is a solution of (MFCQVIP).*
- (b)  *$x \in B, u \in \tilde{P}(x), v \in \tilde{Q}(x)$  and  $G(x) = Q_{K(x)}[G(x) - \tau(T(u) + A(v))]$  for any  $\tau > 0$ .*

*Proof.* It is similar to the proof of Theorem 1 in [3]. □

As in previous sections, we give the following characterization of solutions of (MFCVQIP).

**Lemma 4.4.2.** *Let  $B$  be a real Banach space and  $X$  be a nonempty, closed and convex subset of  $B$ . Let  $T, A, G, m : B \rightarrow B$  be single valued mappings,  $\tilde{P}, \tilde{Q} : B \rightarrow CB(B)$  and  $K : B \rightarrow 2^B$  be multivalued mappings such that  $\forall x \in B, K(x) = m(x) + X$ . Then the set of elements  $x \in B, u \in \tilde{P}(x)$ , and  $v \in \tilde{Q}(x)$  is a solution of (MFCQVIP) if and only if*

$$x = x - G(x) + m(x) + Q_X[G(x) - \tau(T(u) + A(v)) - m(x)], \text{ for any } \tau > 0.$$

Based on the above mentioned observations, we suggest the following iterative algorithm for finding the approximate solutions of (MFCQVIP).

**Algorithm 4.4.1.** *Let  $K(x) = m(x) + X$ , where  $X$  is a nonempty, closed and convex subset of  $B$  and  $\tau > 0$  be fixed. Let  $P, Q : B \rightarrow \mathcal{F}(B)$  be two closed fuzzy mappings satisfying condition  $(*)$  and  $\tilde{P}, \tilde{Q} : B \rightarrow CB(B)$  be multivalued mappings induced by the fuzzy mappings  $P, Q$ , respectively. For given  $x_0 \in B, u_0 \in \tilde{P}(x_0), v_0 \in \tilde{Q}(x_0)$ , we let*

$$x_1 = x_0 - G(x_0) + m(x_0) + Q_X[G(x_0) - \tau(T(u_0) + A(v_0)) - m(x_0)].$$

By Nadler [69], there exist  $u_1 \in \tilde{P}(x_1), v_1 \in \tilde{Q}(x_1)$  such that

$$\|u_0 - u_1\| \leq \tilde{\mathcal{H}}(\tilde{P}(x_0), \tilde{P}(x_1))$$

$$\|v_0 - v_1\| \leq \tilde{\mathcal{H}}(\tilde{Q}(x_0), \tilde{Q}(x_1)).$$

Let

$$x_2 = x_1 - G(x_1) + m(x_1) + Q_X[G(x_1) - \tau(T(u_1) + A(v_1)) - m(x_1)].$$

By induction, we can obtain sequences  $\{x_n\}, \{u_n\}$  and  $\{v_n\}$  satisfying

$$u_n \in \tilde{P}(x_n), \|u_n - u_{n+1}\| \leq \tilde{\mathcal{H}}(\tilde{P}(x_n), \tilde{P}(x_{n+1}))$$

$$v_n \in \tilde{Q}(x_n), \|v_n - v_{n+1}\| \leq \tilde{\mathcal{H}}(\tilde{Q}(x_n), \tilde{Q}(x_{n+1}))$$

and

$$x_{n+1} = x_n - G(x_n) + m(x_n) + Q_X[G(x_n) - \tau(T(u_n) + A(v_n)) - m(x_n)]. \quad (4.4.1)$$

We have the following existence and convergence result.

**Theorem 4.4.3.** *Let  $B$  be a real uniformly smooth Banach space with the module of smoothness  $\tau_B(t) \leq Dt^2$  for some  $D > 0$ . Let  $X$  be a nonempty, closed and convex subset of  $B$ . Let  $P, Q : B \rightarrow \mathcal{F}(B)$  be two closed fuzzy mappings satisfying condition  $(*)$  and  $\tilde{P}, \tilde{Q} : B \rightarrow CB(B)$  be multivalued mappings induced by the fuzzy mappings  $P, Q$ , respectively. Let  $\tilde{P}$  and  $\tilde{Q}$  be  $\tilde{\mathcal{H}}$ -Lipschitz continuous mappings with constants  $\beta$  and  $\eta$ , respectively. Let  $T, A, m : B \rightarrow B$  be Lipschitz continuous with constant  $\alpha, \lambda$  and  $\theta$ , respectively,  $G : B \rightarrow B$  be both strongly accretive with constant  $\gamma > 0$  and Lipschitz continuous with constant  $\delta > 0$ . If*

$$0 < (1 - 2\gamma + 64D\delta^2) + 2\theta + \delta + \tau(\beta\alpha + \lambda\eta) < 1. \quad (4.4.2)$$

*Then there exist  $x \in B, u \in \tilde{P}(x), v \in \tilde{Q}(x)$  such that  $(x, u, v)$  is a solution of (MFCQVIP) and the sequences  $\{x_n\}, \{u_n\}$  and  $\{v_n\}$  generated by the Algorithm 4.4.1 converge strongly to  $x, u$  and  $v$ , respectively, that is,  $x_n \rightarrow x, u_n \rightarrow u$  and  $v_n \rightarrow v$  as  $n \rightarrow \infty$ .*

*Proof.* By (4.4.1), we have

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|x_n - G(x_n) + m(x_n) + Q_X[G(x_n) - \tau(T(u_n) + A(v_n)) \\ &\quad - m(x_n)] - (x_{n-1} - G(x_{n-1}) + m(x_{n-1})) \\ &\quad - Q_X[G(x_{n-1}) - \tau(T(u_{n-1}) + A(v_{n-1})) - m(x_{n-1})]\| \\ &\leq \|x_n - x_{n-1} - (G(x_n) - G(x_{n-1}))\| \\ &\quad + 2\|m(x_n) - m(x_{n-1})\| + \|G(x_n) - G(x_{n-1})\| \\ &\quad + \tau\|T(u_n) - T(u_{n-1})\| + \tau\|A(v_n) - A(v_{n-1})\|. \end{aligned} \quad (4.4.3)$$

From the proof of Theorem 3 in [3], we have

$$\|x_n - x_{n-1} - (G(x_n) - G(x_{n-1}))\|^2 \leq (1 - 2\gamma + 64D\delta^2)\|x_n - x_{n-1}\|^2. \quad (4.4.4)$$

It follows from the Lipschitz property of the corresponding functions that

$$\|m(x_n) - m(x_{n-1})\| \leq \theta\|x_n - x_{n-1}\| \quad (4.4.5)$$

$$\|T(u_n) - T(u_{n-1})\| \leq \beta\alpha\|x_n - x_{n-1}\| \quad (4.4.6)$$

$$\|A(v_n) - A(v_{n-1})\| \leq \lambda\eta\|x_n - x_{n-1}\| \quad (4.4.7)$$

$$\|G(x_n) - G(x_{n-1})\| \leq \delta\|x_n - x_{n-1}\|. \quad (4.4.8)$$

From (4.4.3) - (4.4.8), we have

$$\|x_{n+1} - x_n\| \leq t\|x_n - x_{n-1}\|,$$

where  $t = (1 - 2\gamma + 64D\delta^2) + 2\theta + \delta + \tau(\beta\alpha + \lambda\eta)$  and  $0 < t < 1$  by (4.4.2). Consequently  $\{x_n\}$  is a Cauchy sequence, and thus it converges to some  $x \in B$ . By (4.4.1), we have

$$\|u_n - u_{n-1}\| \leq \tilde{\mathcal{H}}(\tilde{P}(x_n), \tilde{P}(x_{n-1})) \leq \beta\|x_n - x_{n-1}\|,$$

$$\|v_n - v_{n-1}\| \leq \tilde{\mathcal{H}}(\tilde{Q}(x_n), \tilde{Q}(x_{n-1})) \leq \eta\|x_n - x_{n-1}\|,$$

and hence  $\{u_n\}$  and  $\{v_n\}$  are also Cauchy sequences in  $B$ . Let  $\{u_n\}$  and  $\{v_n\}$  converge to some  $u \in B$  and  $v \in B$ , respectively. Since  $Q_X, G, \tilde{P}, \tilde{Q}, T, A$  and  $m$  are all continuous, we have

$$x = x - G(x) + m(x) + Q_X[G(x) - \tau(T(u) + A(v)) - m(x)].$$

Further, we have

$$\begin{aligned} d(u, \tilde{P}(x)) &= \inf\{\|u - z\| : z \in \tilde{P}(x)\} \\ &\leq \|u - u_n\| + d(u_n, \tilde{P}(x)) \\ &\leq \|u - u_n\| + \tilde{\mathcal{H}}(\tilde{P}(x_n), \tilde{P}(x)) \\ &\leq \|u - u_n\| + \beta\|x_n - x\| \rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

and hence  $u \in \tilde{G}(x)$ . Similarly we can show that  $v \in \tilde{D}(x)$ . The result then follows from Lemma 4.4.2.  $\square$

## Chapter 5

# Generalized Nonlinear Variational Inclusions in Banach Spaces

In this chapter, we consider the generalized nonlinear variational inclusion problem (for short, GNVIP) in the setting of Banach spaces. Several special cases of (GNVIP) are also given. By using the resolvent operator technique for  $m$ -accretive operator defined on a Banach space, we convert our problem into a fixed point problem. We use this characterization of a solution of (GNVIP), we propose an iterative algorithm for computing the approximate solutions of (GNVIP). The convergence of approximate solutions obtained by the proposed algorithm and the existence of a solution of (GNVIP) are also studied. In the last section of this chapter, we extend the generalized nonlinear variational inclusion problem for fuzzy mappings. We also extend the iterative algorithm, and convergence and an existence result of second section of this chapter for fuzzy mappings.

### 5.1 Introduction and Formulations

The iterative process for nonlinear equations involving accretive operators and their generalizations has been studied by many authors, for example, Deing [33], Huang [51], Goebel and Reich [39], and references therein. Recently, Huang [49] extend this process for generalized set-valued implicit variational inclusions for  $m$ -accretive operators defined on a Banach space. He first used resolvent technique for  $m$ -accretive

operators to establish the equivalence between the generalized set-valued implicit variational inclusions and the resolvent equations in Banach spaces. Then this equivalence and Nadler's Theorem [69] is used to construct some new iterative algorithms for solving generalized set-valued implicit variational inclusions in real Banach spaces.

In this section we consider a more general inclusion which includes generalized set-valued implicit variational inclusions [49] as a special case. By using the resolvent operator technique for  $m$ -accretive operator defined on a Banach space, we convert our problem into a fixed point problem. We use this characterization of a solution of generalized nonlinear variational inclusion problem (for short, GNVIP), we propose an iterative algorithm for computing the approximate solutions of (GNVIP). The convergence of approximate solutions obtained by the proposed algorithm and the existence of a solution of (GNVIP) are also studied. In the last section of this chapter, we extend the generalized nonlinear variational inclusion problem for fuzzy mappings. We also extend the iterative algorithm, convergence and an existence result of second section of this chapter for fuzzy mappings.

Let  $B$  be a real Banach space and  $B^*$  be its topological dual space. Let  $T, F, P : B \rightarrow CB(B)$  and  $A : B \rightarrow 2^B$  be multivalued mappings,  $N : B \times B \rightarrow B$  and  $g : B \rightarrow B$  be single valued mappings. We consider the following *generalized nonlinear variational inclusion problem*:

$$(GNVIP) \quad \begin{cases} \text{Find } x \in B, u \in P(x), w \in T(x), \text{ and } q \in F(x) \text{ such that} \\ 0 \in u + N(w, q) + A(g(x)). \end{cases}$$

SPECIAL CASES:

(i) If  $P \equiv 0$ , then (GNVIP) is equivalent to find  $x \in B$ ,  $w \in T(x)$ , and  $q \in F(x)$  such that

$$0 \in N(w, q) + A(g(x)), \quad (5.1.1)$$

which is called *set-valued variational inclusion problem*, considered by Chang [20].

(ii) If  $N \equiv 0$ , then (GNVIP) is equivalent to find  $x \in B$  and  $u \in P(x)$  such that

$$0 \in u + A(g(x)), \quad (5.1.2)$$

which is called *generalized set-valued implicit variational inclusion problem* in Banach spaces, introduced and studied by Huang [49]. By using Nadler's Theorem and the resolvent operator technique for  $m$ -accretive mappings in Banach spaces, he constructed some new iterative algorithms for solving this class of generalized set-valued implicit variational inclusions. He studied the existence of solutions and convergence of iterative sequences generated by the algorithm in Banach spaces. An application of such problem is also given by him.

(iii) If  $B = H$  is a Hilbert space and  $A : H \rightarrow 2^H$  is a maximal monotone multivalued mapping, then  $A$  is also an  $m$ -accretive mapping. In this case, generalized set-valued implicit variational inclusion problem is equivalent to find  $x \in H$  and  $u \in P(x)$  such that

$$0 \in u + A(g(x)), \quad (5.1.3)$$

which is called *generalized set-valued implicit variational inclusion problem* in Hilbert spaces and studied by Huang [48].

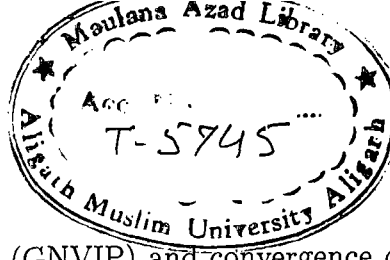
(iv) If  $B = H$  is a Hilbert space and  $A = \partial\phi$ , where  $\phi : H \rightarrow \mathbb{R} \cup \{+\infty\}$  is a proper, convex, lower semicontinuous function on  $H$  and  $\partial\phi$  denotes the subdifferential of function  $\phi$ , then problem (GNVIP) is equivalent to find  $x \in H$ ,  $u \in P(x)$ ,  $w \in T(x)$ , and  $q \in F(x)$  such that

$$\langle u + N(w, q), y - g(x) \rangle \geq \phi(g(x)) - \phi(y), \quad \forall y \in H. \quad (5.1.4)$$

It appears to be new one and a variant form of the problem considered by Huang [45] and Yuan [104].

## 5.2 Existence and Convergence Theory

By using the resolvent operator technique for the  $m$ -accretive mapping in Banach spaces, we convert the generalized nonlinear variational inclusion problem (GNVIP) into a fixed point problem. By using this characterization of solutions of (GNVIP), we propose an iterative algorithm to finding the approximate solutions of (GNVIP). We



study the existence of solutions for (GNVIP) and convergence of iterative sequences generated by the proposed algorithm.

**Definition 5.2.1.** Let  $A : D(A) \subset B \rightarrow 2^B$  be an  $m$ -accretive mapping. For any  $\rho > 0$ , the mapping  $J_\rho^A : B \rightarrow D(A)$  associated with  $A$  defined by

$$J_\rho^A(u) = (I + \rho A)^{-1}(u), \quad u \in B$$

is called *resolvent operator*.

**Definition 5.2.2.** The resolvent operator  $J_\rho^A : B \rightarrow D(A)$  is said to be

(i) *retraction* if  $(I + \rho A)^{-1} \circ (I + \rho A)^{-1}(u) = (I + \rho A)^{-1}(u)$ .

(ii) *nonexpansive retraction* if

$$\|J_\rho^A(z_1) - J_\rho^A(z_2)\| \leq \|z_1 - z_2\|, \quad \forall z_1, z_2 \in B.$$

*Remark 5.2.1.* [14] It is well known that  $J_\rho^A$  is single valued mapping.

**Lemma 5.2.1.** The set  $(x, u, w, q)$  is a solution of (GNVIP) if and only if  $(x, u, w, q)$  satisfies the following relation

$$g(x) = J_\rho^A[g(x) - \rho\{u + N(w, q)\}], \quad (5.2.1)$$

where  $u \in P(x)$ ,  $w \in T(x)$ ,  $q \in F(x)$  and  $\rho > 0$  is a constant.

*Proof.* By the definition of the resolvent operator  $J_\rho^A$  associated with  $A$ , we have that (5.2.1) holds if and only if  $u \in P(x)$ ,  $w \in T(x)$ , and  $q \in F(x)$  such that

$$g(x) - \rho\{u + N(w, q)\} \in g(x) + \rho A(g(x)).$$

The above inclusion holds if and only if  $u \in P(x)$ ,  $w \in T(x)$ , and  $q \in F(x)$  such that

$$0 \in u + N(w, q) + A(g(x)).$$

Hence the set  $(x, u, w, q)$  is a solution of (GNVIP) if and only if  $u \in P(x)$ ,  $w \in T(x)$ , and  $q \in F(x)$  are such that (5.2.1) holds.  $\square$



**Lemma 5.2.2.** [49] *Let  $g : B \rightarrow B$  be a continuous and  $k$ -strongly accretive mapping. Then  $g$  maps  $B$  onto  $B$ .*

We now invoke Lemma 5.2.1, Lemma 5.2.2 and Nadler's Theorem [69] to propose the following iterative algorithm.

**Algorithm 5.2.1.** *Let  $T, F, P : B \rightarrow CB(B)$  be multivalued mappings and  $N : B \times B \rightarrow B$  be a single valued mapping. Let  $g : B \rightarrow B$  is a continuous and  $k$ -strongly accretive mapping. For given  $x_0 \in B$ ,  $u_0 \in P(x_0)$ ,  $w_0 \in T(x_0)$ ,  $q_0 \in F(x_0)$  and  $0 < \epsilon < 1$ , Let*

$$x_1 = x_0 - g(x_0) + J_\rho^A[g(x_0) - \rho\{u_0 + N(w_0, q_0)\}].$$

*Since  $u_0 \in P(x_0) \in CB(B)$ ,  $w_0 \in T(x_0) \in CB(B)$ , and  $q_0 \in F(x_0) \in CB(B)$ , by Nadler [69], there exist  $u_1 \in P(x_1)$ ,  $w_1 \in T(x_1)$ , and  $q_1 \in F(x_1)$  such that*

$$\begin{aligned} \|w_0 - w_1\| &\leq \mathcal{H}(T(x_0), T(x_1)) + \epsilon' \|x_0 - x_1\| \\ \|q_0 - q_1\| &\leq \mathcal{H}(F(x_0), F(x_1)) + \epsilon' \|x_0 - x_1\| \\ \|u_0 - u_1\| &\leq \mathcal{H}(P(x_0), P(x_1)) + \epsilon' \|x_0 - x_1\|. \end{aligned}$$

*Let*

$$x_2 = x_1 - g(x_1) + J_\rho^A[g(x_1) - \rho\{u_1 + N(w_1, q_1)\}].$$

*Continuing the above process inductively, we can obtain the sequences  $\{x_n\}$ ,  $\{u_n\}$ ,  $\{w_n\}$  and  $\{q_n\}$  satisfying*

$$\begin{aligned} x_{n+1} &= x_n - g(x_n) + J_\rho^A[g(x_n) - \rho\{u_n + N(w_n, q_n)\}], \quad n = 0, 1, 2, \dots \\ u_n &\in P(x_n), \quad \|u_n - u_{n+1}\| \leq \mathcal{H}(P(x_n), P(x_{n+1})) + \epsilon^{n+1} \|x_n - x_{n+1}\| \\ w_n &\in T(x_n), \quad \|w_n - w_{n+1}\| \leq \mathcal{H}(T(x_n), T(x_{n+1})) + \epsilon^{n+1} \|x_n - x_{n+1}\| \\ q_n &\in F(x_n), \quad \|q_n - q_{n+1}\| \leq \mathcal{H}(F(x_n), F(x_{n+1})) + \epsilon^{n+1} \|x_n - x_{n+1}\|. \end{aligned} \quad (5.2.2)$$

Now we prove the existence of solutions of (GNVIP) and the convergence of sequences  $\{x_n\}$ ,  $\{u_n\}$ ,  $\{w_n\}$ , and  $\{q_n\}$  obtained by Algorithm 5.2.1.

**Theorem 5.2.3.** *Let  $B$  be a real uniformly smooth Banach space with the module of smoothness  $\tau_\beta(t) \leq Ct^2$  for some  $C > 0$ . Let  $T, F, P : B \rightarrow CB(B)$  be multivalued mappings such that  $T$  is  $\mathcal{H}$ -Lipschitz continuous with constant  $\mu$ ,  $F$  is  $\mathcal{H}$ -Lipschitz continuous with constant  $t$ ,  $P$  is  $\mathcal{H}$ -Lipschitz continuous with constant  $\gamma$ . Let  $g : B \rightarrow B$  be strongly accretive with constant  $\gamma$  and Lipschitz continuous with constant  $\delta$ , and  $N : B \times B \rightarrow B$  be Lipschitz continuous in both the arguments with constants  $\alpha$  and  $\beta$ , respectively. If  $(I \circ \rho A)^{-1} \circ (I \circ \rho A)^{-1} = (I + \rho A)^{-1}$  and*

$$0 < (1 - 2\gamma + 64C\delta^2)^{1/2} + [\delta - \rho(\alpha\mu + \beta t + \gamma)] < 1, \quad (5.2.3)$$

*then there exist  $x \in B$ ,  $w \in T(x)$ ,  $q \in F(x)$ , and  $u \in P(x)$  such that the set  $(x, u, w, q)$  is a solution of (GNVIP) and the iterative sequences  $\{x_n\}$ ,  $\{w_n\}$ ,  $\{q_n\}$  and  $\{u_n\}$  generated by Algorithm 5.2.1 converge strongly to  $x, w, q$ , and  $u$  in  $B$ , respectively.*

*Proof.* From Algorithm 5.2.1, we have

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|x_n - g(x_n) + J_\rho^A(g(x_n) - \rho\{u_n + N(w_n, q_n)\}) \\ &\quad - [x_{n-1} - g(x_{n-1}) + J_\rho^A(g(x_{n-1}) - \rho\{u_{n-1} + N(w_{n-1}, q_{n-1})\})]\| \\ &\leq \|x_n - x_{n-1} - (g(x_n) - g(x_{n-1}))\| + \|J_\rho^A(z_n) - J_\rho^A(z_{n-1})\|, \end{aligned} \quad (5.2.4)$$

where  $z_n = g(x_n) - \rho\{u_n + N(w_n, q_n)\}$ .

By Proposition 1.2.8, we have, (see, Alber and Yao [3])

$$\|x_n - x_{n-1} - (g(x_n) - g(x_{n-1}))\|^2 \leq (1 - 2\gamma + 64C\delta^2)\|x_n - x_{n-1}\|^2,$$

and as  $J_\rho^A$  is nonexpansive, we have

$$\begin{aligned} \|J_\rho^A(z_n) - J_\rho^A(z_{n-1})\| &\leq \|z_n - z_{n-1}\| \\ &= \|g(x_n) - \rho\{u_n + N(w_n, q_n)\} - (g(x_{n-1}) - \rho\{u_{n-1} + N(w_{n-1}, q_{n-1})\})\| \\ &= \|g(x_n) - g(x_{n-1})\| - \rho\|N(w_n, q_n) - N(w_{n-1}, q_{n-1})\| - \rho\|u_n - u_{n-1}\| \\ &= \|g(x_n) - g(x_{n-1})\| - \rho\|N(w_n, q_n) - N(w_{n-1}, q_n) + N(w_{n-1}, q_n) \\ &\quad - N(w_{n-1}, q_{n-1})\| - \rho\|u_n - u_{n-1}\| \\ &\leq \|g(x_n) - g(x_{n-1})\| - \rho\|N(w_n, q_n) - N(w_{n-1}, q_n)\| \\ &\quad - \rho\|N(w_{n-1}, q_n) - N(w_{n-1}, q_{n-1})\| - \rho\|u_n - u_{n-1}\|. \end{aligned} \quad (5.2.5)$$

Using the Lipschitz continuity of  $N(., .)$  with respect to first argument and  $\mathcal{H}$ -Lipschitz continuity of  $T$ , we have

$$\begin{aligned} \|N(w_n, q_n) - N(w_{n-1}, q_n)\| &\leq \alpha \|w_n - w_{n-1}\| \\ &\leq \alpha \mathcal{H}(T(x_n), T(x_{n-1})) + \epsilon^n \|x_n - x_{n-1}\| \\ &\leq \alpha(\mu + \epsilon^n) \|x_n - x_{n-1}\|. \end{aligned} \quad (5.2.6)$$

In a similar way,

$$\begin{aligned} \|N(w_{n-1}, q_n) - N(w_{n-1}, q_{n-1})\| &\leq \beta \|q_n - q_{n-1}\| \\ &\leq \beta \mathcal{H}(F(x_n), F(x_{n-1})) + \epsilon^n \|x_n - x_{n-1}\| \\ &\leq \beta(t + \epsilon^n) \|x_n - x_{n-1}\|. \end{aligned} \quad (5.2.7)$$

Using (5.2.6) - (5.2.7), (5.2.5) can be written as

$$\begin{aligned} \|J_\rho^A(z_n) - J_\rho^A(z_{n-1})\| &\leq \delta \|x_n - x_{n-1}\| - \rho\alpha(\mu + \epsilon^n) \|x_n - x_{n-1}\| \\ &\quad - \rho\beta(t + \epsilon^n) \|x_n - x_{n-1}\| - \rho(\gamma + \epsilon^n) \|x_n - x_{n-1}\| \\ &= [\delta - \rho\alpha(\mu + \epsilon^n) - \rho\beta(t + \epsilon^n) - \rho(\gamma + \epsilon^n)] \|x_n - x_{n-1}\|. \end{aligned}$$

Thus we have

$$\|x_{n+1} - x_n\| \leq \theta(\epsilon^n) \|x_n - x_{n-1}\|,$$

where  $\theta(\epsilon^n) = (1 - 2\gamma + 64C\delta^2)^{1/2} + [\delta - \rho(\alpha\mu + \beta t + \gamma) - \rho(\alpha + t + 1)\epsilon^n]$ .

Let  $\theta = (1 - 2\gamma + 64C\delta^2)^{1/2} + [\delta - \rho(\alpha\mu + \beta t + \gamma)]$ . Since  $0 < \epsilon < 1$ , it follows that

$$\theta(\epsilon^n) \rightarrow \theta, \text{ as } n \rightarrow \infty.$$

From (5.2.3), we have  $\theta < 1$ , and consequently the sequence  $\{x_n\}$  is a Cauchy sequence in  $B$ . Since  $B$  is a Banach space, there exist  $x \in B$  such that  $x_n \rightarrow x$  as  $n \rightarrow \infty$ .

From (5.2.5) and (5.2.6), we see that  $w_n$  and  $q_n$  are Cauchy sequences in  $B$ , that is, there exist  $w$  and  $q \in B$  such that  $w_n \rightarrow w$  and  $q_n \rightarrow q$ . Now using the continuity of the operators  $T, F, P, g, N, J_\rho^A$  and Algorithm 5.2.1, we have

$$x = x - g(x) + J_\rho^A[g(x) - \rho\{u + N(w, q)\}].$$

Finally, we prove that  $w \in T(x)$ ,  $q \in F(x)$ , and  $u \in P(x)$ . In fact, since  $w \in T(x_n)$ , we have

$$\begin{aligned} d(w, T(x)) &\leq \|w - w_n\| + d(w_n, T(x)) \\ &\leq \|w - w_n\| + \mathcal{H}(T(x_n), T(x)) \\ &\leq \|w - w_n\| + \mu \|x_n - x\| \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

which implies that  $d(w, T(x)) = 0$  and hence  $w \in T(x)$ . Similarly we can prove that  $q \in F(x)$  and  $u \in P(x)$ . Then by Lemma 5.2.1, the result follows.  $\square$

### 5.3 Fuzzy Extension

In this section, we consider the generalized variational inclusion problem for fuzzy mappings and extend the results and algorithm of previous section for fuzzy mappings.

Let  $N : B \times B \rightarrow B$  and  $g : B \rightarrow B$  be the single valued mappings and let  $T, F, P : B \rightarrow \mathcal{F}(B)$  be fuzzy mappings. Let  $a, b, c : B \rightarrow [0, 1]$  be given functions. Suppose that  $A : B \rightarrow 2^B$  is an  $m$ -accretive mapping. We consider the following problem:

$$(GNVIPFM) \quad \begin{cases} \text{Find } x, w, q, \text{ and } u \in B \text{ such that} \\ T_x(w) \geq a(x), F_x(q) \geq b(x), P_x(u) \geq c(x), \text{ and} \\ 0 \in u + N(w, q) + A(g(x)). \end{cases}$$

It is called *generalized nonlinear variational inclusion problem for fuzzy mappings* in Banach spaces.

The (GNVIPFM) includes many known problems studied in recent past, see for example [46] and references therein.

As in previous section, we first transfer (GNVIPFM) into a fixed point problem.

**Theorem 5.3.1.** *The set  $(x, w, q, u)$  is a solution of (GNVIPFM) if and only if  $(x, w, q, u)$  satisfies the following relations:*

$$g(x) = J_\rho^A[g(x) - \rho\{u + N(w, q)\}], \quad (5.3.1)$$

where  $w \in \tilde{T}(x)$ ,  $q \in \tilde{F}(x)$ ,  $u \in \tilde{P}(x)$  and  $\rho > 0$  is a constant.

*Proof.* By the definition of the resolvent operator  $J_\rho^A$  associated with  $A$ , we have that (5.3.1) holds if and only if  $w \in \tilde{T}(x)$ ,  $q \in \tilde{F}(x)$ ,  $u \in \tilde{P}(x)$  such that

$$g(x) - \rho\{u + N(w, q)\} \in g(x) + \rho A(g(x))$$

The above inclusion holds if and only if  $w \in \tilde{T}(x)$ ,  $q \in \tilde{F}(x)$  and  $u \in \tilde{P}(x)$  such that

$$0 \in u + N(w, q) + A(g(x)).$$

Hence  $(x, w, q, u)$  is a solution of (GNVIPFM) if and only if  $w \in \tilde{T}(x)$ ,  $q \in \tilde{F}(x)$ , and  $u \in \tilde{P}(x)$  are such that (5.3.1) holds.  $\square$

The following algorithm is an extension of Algorithm 5.2.1 for fuzzy mappings for computing the approximate solutions of (GNVIPFM).

**Algorithm 5.3.1.** *Let  $T, F, P : B \rightarrow \mathcal{F}(B)$  be closed fuzzy mappings satisfying condition (\*), of section 4.4 and  $\tilde{T}, \tilde{F}, \tilde{P} : B \rightarrow CB(B)$  be multivalued mappings induced by the fuzzy mappings  $T, F$ , and  $P$ , respectively. Let  $N : B \times B \rightarrow B$  be a single valued bifunction and  $g : B \rightarrow B$  be a continuous and  $k$ -strongly accretive mapping. For given  $x_0 \in B$ ,  $w_0 \in \tilde{T}(x_0)$ ,  $q_0 \in \tilde{F}(x_0)$ , and  $u_0 \in \tilde{P}(x_0)$ , we let*

$$g(x_1) = J_\rho^A[g(x_0) - \rho\{u_0 + N(w_0, q_0)\}].$$

*Since  $w_0 \in \tilde{T}(x_0) \in CB(B)$ ,  $q_0 \in \tilde{F}(x_0) \in CB(B)$ , and  $u_0 \in \tilde{P}(x_0) \in CB(B)$ , by Nadler [69], there exist  $w_1 \in \tilde{T}(x_1)$ ,  $q_1 \in \tilde{F}(x_1)$ , and  $u_1 \in \tilde{P}(x_1)$  such that*

$$\begin{aligned} \|w_0 - w_1\| &\leq (1 + 1)\tilde{\mathcal{H}}(\tilde{T}(x_0), \tilde{T}(x_1)), \\ \|q_0 - q_1\| &\leq (1 + 1)\tilde{\mathcal{H}}(\tilde{F}(x_0), \tilde{F}(x_1)), \\ \|u_0 - u_1\| &\leq (1 + 1)\tilde{\mathcal{H}}(\tilde{P}(x_0), \tilde{P}(x_1)). \end{aligned}$$

*Let*

$$g(x_2) = J_\rho^A[g(x_1) - \rho\{u_1 + N(w_1, q_1)\}].$$

*Continuing the above process inductively, we can obtain sequences  $\{x_n\}$ ,  $\{w_n\}$ ,  $\{q_n\}$  and  $\{u_n\}$  satisfying*

$$g(x_{n+1}) = J_\rho^A[g(x_n) - \rho\{u_n + N(w_n, q_n)\}],$$

$$\begin{aligned} w_n \in \tilde{T}(x_n), \quad \|w_n - w_{n+1}\| &\leq (1 + \frac{1}{n+1})\tilde{\mathcal{H}}(\tilde{T}(x_n), \tilde{T}(x_{n+1})), \\ q_n \in \tilde{F}(x_n), \quad \|q_n - q_{n+1}\| &\leq (1 + \frac{1}{n+1})\tilde{\mathcal{H}}(\tilde{F}(x_n), \tilde{F}(x_{n+1})), \\ u_n \in \tilde{P}(x_n), \quad \|u_n - u_{n+1}\| &\leq (1 + \frac{1}{n+1})\tilde{\mathcal{H}}(\tilde{P}(x_n), \tilde{P}(x_{n+1})). \end{aligned}$$

$n = 0, 1, 2, \dots$

Now we prove the existence of solutions for (GNVIPFM) and the convergence of iterative sequences generated by Algorithm 5.3.1.

**Theorem 5.3.2.** *Let  $B$  be real Banach space. Let  $T, F, P : B \rightarrow \mathcal{F}(B)$  be closed fuzzy mappings satisfying condition  $(*)$ , of section 4.4 and  $\tilde{T}, \tilde{F}, \tilde{P} : B \rightarrow CB(B)$  be multivalued mappings induced by the fuzzy mappings  $T, F$ , and  $P$ , respectively. Let  $A : B \rightarrow 2^B$  be an  $m$ -accretive mapping and  $g : B \rightarrow B$  be  $\sigma$ -Lipschitz continuous and  $k$ -strongly accretive. Assume that  $\tilde{T}$  is  $\mu$ -Lipschitz continuous,  $\tilde{F}$  is  $\gamma$ -Lipschitz continuous,  $\tilde{P}$  is  $t$ -Lipschitz continuous and  $N : B \times B \rightarrow B$  is Lipschitz continuous in both the arguments with constant  $\alpha, \beta$ , respectively. If*

$$2k - 1 - \sigma^2 < \rho(t + \beta\gamma + \alpha\mu)\sqrt{2k - 1},$$

$$k > \frac{1 + \sigma^2}{2} \quad (5.3.2)$$

*then there exist  $x \in B$ ,  $w \in \tilde{T}(x)$ ,  $q \in \tilde{F}(x)$ , and  $u \in \tilde{P}(x)$  such that the set  $(x, w, q, u)$  is a solution of (GNVIPFM) and  $x_n \rightarrow x$ ,  $w_n \rightarrow w$ ,  $q_n \rightarrow q$ , and  $u_n \rightarrow u$  as  $n \rightarrow \infty$ .*

*Proof.* Since  $\tilde{T}$  is  $\mu$ -Lipschitz continuous,  $\tilde{F}$  is  $\gamma$ -Lipschitz continuous and  $\tilde{P}$  is  $t$ -Lipschitz continuous, it follows from Algorithm 5.3.1 that

$$\begin{aligned} \|w_n - w_{n+1}\| &\leq \left(1 + \frac{1}{n+1}\right) \mathcal{H}(\tilde{T}(x_n), \tilde{T}(x_{n+1})) \leq \left(1 + \frac{1}{n+1}\right) \mu \|x_n - x_{n+1}\|, \\ \|q_n - q_{n+1}\| &\leq \left(1 + \frac{1}{n+1}\right) \mathcal{H}(\tilde{F}(x_n), \tilde{F}(x_{n+1})) \leq \left(1 + \frac{1}{n+1}\right) \gamma \|x_n - x_{n+1}\|, \\ \|u_n - u_{n+1}\| &\leq \left(1 + \frac{1}{n+1}\right) \mathcal{H}(\tilde{P}(x_n), \tilde{P}(x_{n+1})) \leq \left(1 + \frac{1}{n+1}\right) t \|x_n - x_{n+1}\|. \end{aligned}$$

$$n = 0, 1, 2, \dots \quad (5.3.3)$$

Let

$$z_{n+1} = g(x_n) - \rho\{u_n + N(w_n, q_n)\}, \quad n = 0, 1, 2, \dots$$

Since  $g$  is  $\sigma$ -Lipschitz continuous,  $N(., .)$  is Lipschitz continuous in both the arguments

with constants  $\alpha$  and  $\beta$ , respectively, and using Lemma 5.3.2 and (5.3.3), we have

$$\begin{aligned}
\|z_{n+1} - z_n\|^2 &= \|g(x_n) - g(x_{n-1}) - \rho[u_n + N(w_n, q_n) - (u_{n-1} + N(w_{n-1}, q_{n-1}))]\|^2 \\
&\leq \|g(x_n) - g(x_{n-1})\|^2 - 2\rho\langle u_n + N(w_n, q_n) - (u_{n-1} + N(w_{n-1}, q_{n-1})), j(z_{n+1} - z_n) \rangle \\
&\leq \sigma^2\|x_n - x_{n-1}\|^2 + 2\rho\|u_n + N(w_n, q_n) - (u_{n-1} + N(w_{n-1}, q_{n-1}))\| \\
&\quad \|z_{n+1} - z_n\| \\
&\leq \sigma^2\|x_n - x_{n-1}\|^2 + 2\rho[\|u_n - u_{n-1}\| + \|N(w_n, q_n) - N(w_n, q_{n-1}) \\
&\quad + N(w_n, q_{n-1}) - N(w_{n-1}, q_{n-1})\|]\|z_{n+1} - z_n\| \\
&\leq \sigma^2\|x_n - x_{n-1}\|^2 + 2\rho[\|u_n - u_{n-1}\| + \|N(w_n, q_n) - N(w_n, q_{n-1})\| \\
&\quad + \|N(w_n, q_{n-1}) - N(w_{n-1}, q_{n-1})\|]\|z_{n+1} - z_n\| \\
&\leq \sigma^2\|x_n - x_{n-1}\|^2 + 2\rho[\|u_n - u_{n-1}\| + \beta\|q_n - q_{n-1}\| + \alpha\|w_n - w_{n-1}\|] \\
&\quad \|z_{n+1} - z_n\| \\
&\leq \sigma^2\|x_n - x_{n-1}\|^2 + 2\rho[t(1 + \frac{1}{n}) + \beta\gamma(1 + \frac{1}{n}) + \alpha\mu(1 + \frac{1}{n})] \\
&\quad \|x_n - x_{n-1}\|\|z_{n+1} - z_n\| \\
&= \sigma^2\|x_n - x_{n-1}\|^2 + 2\rho(1 + \frac{1}{n})(t + \beta\gamma + \alpha\mu) \\
&\quad \|x_n - x_{n-1}\|\|z_{n+1} - z_n\|. \tag{5.3.4}
\end{aligned}$$

Since  $g$  is  $k$ -strongly accretive,  $J_\rho^A$  is nonexpansive and from Proposition 1.2.8, it follows that,

$$\begin{aligned}
\|x_n - x_{n-1}\|^2 &= \|[J_\rho^A(z_n) - J_\rho^A(z_{n-1})] - [g(x_n) - x_n - (g(x_{n-1}) - x_{n-1})]\|^2 \\
&\leq \|J_\rho^A(z_n) - J_\rho^A(z_{n-1})\|^2 \\
&\quad - 2\langle g(x_n) - x_n - (g(x_{n-1}) - x_{n-1}), j(x_n - x_{n-1}) \rangle \\
&\leq \|z_n - z_{n-1}\|^2 - 2k\|x_n - x_{n-1}\|^2 + 2\|x_n - x_{n-1}\|^2,
\end{aligned}$$

which implies that

$$\|x_n - x_{n-1}\| \leq \frac{1}{\sqrt{2k-1}}\|z_n - z_{n-1}\|. \tag{5.3.5}$$

From (5.3.4) and (5.3.5), we have

$$\begin{aligned}
\|z_n - z_{n-1}\|^2 &\leq \frac{\sigma^2}{2k-1} \|z_n - z_{n-1}\|^2 \\
&\quad + \left[ \frac{2\rho(1+1/n)(t+\beta\gamma+\alpha\mu)}{\sqrt{2k-1}} \right] \|z_n - z_{n-1}\| \|z_{n+1} - z_n\| \\
&\leq \frac{\sigma^2}{2k-1} \|z_n - z_{n-1}\|^2 \\
&\quad + \left[ \frac{\rho(1+1/n)(t+\beta\gamma+\alpha\mu)}{\sqrt{2k-1}} \right] \|z_n - z_{n-1}\|^2 + \|z_{n+1} - z_n\|^2.
\end{aligned}$$

Finally, we have

$$\|z_{n+1} - z_n\| \leq \theta_n \|z_n - z_{n-1}\| \quad (5.3.6)$$

where

$$\theta_n = \frac{\sigma^2 + \rho(1+1/n)(t+\beta\gamma+\alpha\mu)\sqrt{2k-1}}{(2k-1)[1 - \rho(1+1/n)(t+\beta\gamma+\alpha\mu)/\sqrt{2k-1}]}.$$

Letting

$$\theta = \frac{\sigma^2 + \rho(t+\beta\gamma+\alpha\mu)\sqrt{2k-1}}{(2k-1)[1 - \rho(t+\beta\gamma+\alpha\mu)/\sqrt{2k-1}]}.$$

We know that  $\theta_n \rightarrow \theta$  as  $n \rightarrow \infty$ . From condition (5.3.2), it follows that  $\theta < 1$ . Hence  $\theta_n < 1$  for  $n$  sufficiently large. Therefore (5.3.6) implies that  $\{z_n\}$  is a Cauchy sequence in  $B$ . Since  $B$  is a Banach space, there exists  $z \in B$  such that  $z_n \rightarrow z$  as  $n \rightarrow \infty$ . From (5.3.3) and (5.3.5), we can easily verify that  $\{x_n\}$ ,  $\{w_n\}$ ,  $\{q_n\}$  and  $\{u_n\}$  are also Cauchy sequences in  $B$ . Therefore, there exist  $x \in B$ ,  $w \in B$ ,  $q \in B$ , and  $u \in B$  such that  $x_n \rightarrow x$ ,  $w_n \rightarrow w$ ,  $q_n \rightarrow q$ , and  $u_n \rightarrow u$  as  $n \rightarrow \infty$ . Note that  $w_n \in \tilde{T}(x_n)$ , we have

$$\begin{aligned}
d(w, \tilde{T}(x)) &= \inf\{\|w_n - P\| : P \in \tilde{T}(x)\} \\
&\leq \|w - w_n\| + d(w_n, \tilde{T}(x)) \\
&\leq \|w - w_n\| + \tilde{\mathcal{H}}(\tilde{T}(x_n), \tilde{T}(x)) \\
&\leq \|w - w_n\| + \mu\|x_n - x\| \rightarrow 0, \text{ as } n \rightarrow \infty,
\end{aligned}$$

which implies that  $d(w, \tilde{T}(x)) = 0$ . Since  $\tilde{T}(x) \in CB(B)$ , it follows that  $w \in \tilde{T}(x)$ . Similarly we can show that  $q \in \tilde{F}(x)$  and  $u \in \tilde{P}(x)$ . Since  $g$ ,  $J_\rho^A$ ,  $N(\cdot, \cdot)$ ,  $\tilde{T}$ ,  $\tilde{F}$  and  $\tilde{P}$  are continuous, it follows from Algorithm 5.3.1 that

$$g(x) = J_\rho^A[g(x) - \rho\{u + N(w, q)\}].$$

By Theorem 5.3.1, the set  $(x, w, q, u)$  is a solution of (GNVIPFM).  $\square$



## Chapter 6

# Generalized Quasi-complementarity Problems with Fuzzy Multivalued Maps

In this chapter, we study a class of generalized quasi-complementarity problems with fuzzy multivalued mappings and suggest a new algorithm for computing the approximate solutions of this class of generalized quasi-complementarity problems. We also discuss the existence of a solution of our problem without compactness assumption and the convergence of the iterative sequences generated by the algorithm. Some special cases are also given.

### 6.1 Introduction and Formulations

In an unpublished work, Ansari [4] introduced the concept of variational inequalities for fuzzy mappings in his Ph.D. thesis. Separately, Chang and Zhu [23] also studied a class of variational inequalities for fuzzy mappings. Several kind of variational inequalities and complementarity problems for fuzzy mappings were considered and studied by Chang [18], Chang and Huang [22], Noor [73] and Lee et al [62]. For a detail on Complementarity problems and their applications, we refer to Isac [57]. Motivated and inspired by the recent research work going on in this field, in this chapter, we study a class of generalized quasi-complementarity problems with fuzzy

multivalued mappings. A new algorithm for computing the approximate solutions of the generalized quasi-complementarity problem with fuzzy multivalued mappings is suggested. We also discuss the existence of a solution of our problem without compactness assumption and the convergence of iterative sequences generated by our algorithm.

Let  $H$  be a real Hilbert space endowed with the norm  $\|\cdot\|$  and inner product  $\langle \cdot, \cdot \rangle$ . If  $K$  is a closed convex cone in  $H$ , we denote by  $K^*$  the polar cone of  $K$ , i.e.,

$$K^* = \{u \in H : \langle u, v \rangle \geq 0, \quad \forall v \in K\}.$$

Let  $N : H \times H \rightarrow H$  and  $m, g : H \rightarrow H$  be single valued mappings and  $F, G, A : H \rightarrow \mathcal{F}(H)$  be fuzzy mappings. Let  $a, b, c : H \rightarrow [0, 1]$  be given functions. We consider the following *generalized quasi-complementarity problem with fuzzy multivalued mappings*:

$$(GQCPFM) \quad \begin{cases} \text{Find } u, x, y, w \in H \text{ such that} \\ F_u(x) \geq a(u), \quad G_u(y) \geq b(u), \quad A_u(w) \geq c(u), \\ g(u) \in K(w), \quad N(x, y) \in K^*(w) \text{ and} \\ \langle g(u) - m(w), N(x, y) \rangle = 0, \end{cases}$$

where  $K(w) = m(w) + K$  and thus  $K^*(w) = (m(w) + K)^* = m^*(w) \cap K^*$ .

SPECIAL CASES:

(i) If  $F, G, A : H \rightarrow CB(H)$  are classical multivalued mappings, we can define the fuzzy mappings  $F, G, A : H \rightarrow \mathcal{F}(H)$  by

$$u \rightarrow \chi F(u), \quad u \rightarrow \chi G(u), \quad u \rightarrow \chi A(u),$$

where  $\chi F(u)$ ,  $\chi G(u)$ ,  $\chi A(u)$  are the characteristic functions of  $F(u)$ ,  $G(u)$  and  $A(u)$ , respectively. Taking  $a(u) = b(u) = c(u) = 1$ ,  $\forall u \in H$ , (GQCPFM) is equivalent to the following problem:

$$\begin{cases} \text{Find } u \in H, x \in F(u), y \in G(u), w \in A(u) \text{ such that} \\ g(u) \in K(w), \quad N(x, y) \in K^*(w) \text{ and} \\ \langle g(u) - m(w), N(x, y) \rangle = 0. \end{cases} \quad (6.1.1)$$

Problem (6.1.1) is a generalization of the problem considered in [45].

(ii) If  $N(x, y) = x - y$ ,  $\forall x, y \in H$  and  $A(u) = u$ , then the problem (6.1.1) is equivalent to the following *generalized strongly nonlinear quasi-complementarity problem*:

$$(GSNQCP) \quad \begin{cases} \text{Find } u \in H, x \in F(u), y \in G(u) \text{ such that} \\ g(u) \in K(u), x - y \in K^*(u) \text{ and} \\ \langle g(u) - m(u), x - y \rangle = 0. \end{cases}$$

It is considered and studied by Deing [34].

## 6.2 Existence and Convergence Theory

In this section, we first convert (GQCPFM) into a quasi-variational inequality problem for fuzzy mappings. By using this equivalence and projection operator, we suggest an iterative algorithm for finding the approximate solutions of (GQCPFM). The convergence of approximate solutions obtained by the suggested algorithm is also studied.

**Theorem 6.2.1.** *If  $K \subset H$  is a closed convex cone and  $K(w) = m(w) + K$ , then the set of elements  $(u, x, y, w)$  is a solution of (GQCPFM) if and only if  $u \in H, x \in \tilde{F}(u), y \in \tilde{G}(u)$  and  $w \in \tilde{A}(u)$  such that*

$$g(u) \in K(w) : \langle N(x, y), g(v) - g(u) \rangle \geq 0, \forall g(v) \in K(w). \quad (6.2.1)$$

*Proof.* Let the set of elements  $u \in H, x \in \tilde{F}(u), y \in \tilde{G}(u)$  and  $w \in \tilde{A}(u)$  is a solution of (GQCPFM). Since  $K(w) = m(w) + K$ , if  $g(v) \in K(w)$ , it can be written as  $g(v) = m(w) + z$  for some  $z \in K$  and thus  $\forall g(v) \in K(w)$ , we have

$$\begin{aligned} \langle N(x, y), g(v) - g(u) \rangle &= \langle N(x, y), m(w) + z - g(u) \rangle \\ &= \langle N(x, y), m(w) - g(u) \rangle + \langle N(x, y), z \rangle \\ &= \langle N(x, y), z \rangle \geq 0. \end{aligned}$$

Therefore  $(u, x, y, w)$  is a solution of problem (6.2.1)

Conversely, Suppose that the set of elements  $u \in H, x \in \tilde{F}(u), y \in \tilde{G}(u)$  and

$w \in \tilde{A}(u)$  such that  $g(u) \in K(w)$  satisfies the inequality (6.2.1). Since  $g(u) \in K(w)$ , we know that  $g(u) - m(w) \in K$  and hence  $2g(u) - m(w) \in K(w)$ . From  $0 \in K$ , we get  $m(w) \in K(w)$ . Taking  $g(v) = 2g(u) - m(w)$  and  $g(v) = m(w)$  in (6.2.1), we obtain

$$\langle N(x, y), g(u) - m(w) \rangle \geq 0 \quad \text{and} \quad \langle N(x, y), m(w) - g(u) \rangle \geq 0.$$

It follows from the above inequalities that

$$\langle N(x, y), g(u) - m(w) \rangle = 0$$

It remains to prove that  $N(x, y) \in K^*(w)$ . Taking  $g(v) = m(w) + z$  in (6.2.1), we obtain

$$\begin{aligned} 0 \leq \langle N(x, y), g(v) - g(u) \rangle &= \langle N(x, y), m(w) + z - g(u) \rangle \\ &= \langle N(x, y), z \rangle. \end{aligned}$$

This implies that  $N(x, y) \in K^*(w)$ . Therefore  $(u, x, y, w)$  is a solution of (GQCPFM).  $\square$

We need the following lemma whose proof is similar to the proof of Theorem 3.1 in [71].

**Lemma 6.2.2.** *If  $K$  is a closed convex cone in  $H$  and  $K(w) = m(w) + K$ ,  $\forall w \in K$ . Then  $u \in H, x \in \tilde{F}(u), y \in \tilde{G}(u)$  and  $w \in \tilde{A}(u)$  satisfy (6.2.1) if and only if these satisfy the relation*

$$g(u) = m(w) + P_{K(w)}[g(u) - m(w) - \rho N(x, y)],$$

where  $\rho > 0$  is a constant.

On the basis of Theorem 6.2.1 and Lemma 6.2.2, we propose the following algorithm.

**Algorithm 6.2.1.** *Suppose that  $K$  is a closed convex cone in  $H$  and  $N : H \times H \rightarrow H$ ,  $m, g : H \rightarrow H$ . Let  $F, G, A : H \rightarrow \mathcal{F}(H)$  be fuzzy mappings satisfying condition (\*) of section 4.4 and  $\tilde{F}, \tilde{G}, \tilde{A} : H \rightarrow CB(H)$  be fuzzy mappings induced by  $F, G$  and  $A$ , respectively. For given  $u_0 \in H$ , we take  $x_0 \in \tilde{F}(u_0)$ ,  $y_0 \in \tilde{G}(u_0)$  and  $w_0 \in \tilde{A}(u_0)$  and let*

$$g(u_1) = m(w_0) + P_{K(w_0)}[g(u_0) - m(w_0) - \rho N(x_0, y_0)],$$

where  $\rho > 0$  is a constant.

Since  $x_0 \in \tilde{F}(u_0) \in CB(H)$ ,  $y_0 \in \tilde{G}(u_0) \in CB(H)$  and  $w_0 \in \tilde{A}(u_0) \in CB(H)$ , by Nadler [69], there exist  $x_1 \in \tilde{F}(u_1)$ ,  $y_1 \in \tilde{G}(u_1)$  and  $w_1 \in \tilde{A}(u_1)$  such that

$$\begin{aligned}\|x_0 - x_1\| &\leq (1 + 1)\hat{\mathcal{H}}(\tilde{F}(u_0), \tilde{F}(u_1)), \\ \|y_0 - y_1\| &\leq (1 + 1)\hat{\mathcal{H}}(\tilde{G}(u_0), \tilde{G}(u_1)), \\ \|w_0 - w_1\| &\leq (1 + 1)\hat{\mathcal{H}}(\tilde{A}(u_0), \tilde{A}(u_1)).\end{aligned}$$

Let

$$g(u_2) = m(w_1) + P_{K(w_1)}[g(u_1) - m(w_1) - \rho N(x_1, y_1)].$$

By induction, we can obtain sequences  $\{x_n\}$ ,  $\{y_n\}$ ,  $\{w_n\}$  and  $\{u_n\}$  such that

$$\begin{aligned}x_n \in \tilde{F}(u_n), \|x_n - x_{n+1}\| &\leq (1 + (1 + n)^{-1})\hat{\mathcal{H}}(\tilde{F}(u_n), \tilde{F}(u_{n+1})), \\ y_n \in \tilde{G}(u_n), \|y_n - y_{n+1}\| &\leq (1 + (1 + n)^{-1})\hat{\mathcal{H}}(\tilde{G}(u_n), \tilde{G}(u_{n+1})), \\ w_n \in \tilde{A}(u_n), \|w_n - w_{n+1}\| &\leq (1 + (1 + n)^{-1})\hat{\mathcal{H}}(\tilde{A}(u_n), \tilde{A}(u_{n+1})),\end{aligned}$$

$$g(u_{n+1}) = m(w_n) + P_{K(w_n)}[g(u_n) - m(w_n) - \rho N(x_n, y_n)], \quad n = 0, 1, 2, 3, \dots$$

where  $\rho > 0$  is a constant.

We consider those conditions under which the solution of (GQCPFM) exists and the sequences of the approximate solutions obtained by Algorithm 6.2.1 converge strongly to the exact solution of (GQCPFM).

**Theorem 6.2.3.** *Let  $K$  is a closed convex cone in  $H$ ,  $F, G, A : H \rightarrow \mathcal{F}(H)$  be fuzzy mappings satisfying the condition  $(*)$  of section 4.4 and let  $\tilde{F}, \tilde{G}, \tilde{A} : H \rightarrow CB(H)$  be multivalued mappings induced by  $F, G$  and  $A$ , respectively. Let  $m, g : H \rightarrow H$  be single valued mappings such that  $g(H)$  is closed in  $H$  and  $m$  is Lipschitz continuous with constant  $t$ . Let  $\tilde{F}, \tilde{G}$  and  $\tilde{A}$  are  $\hat{\mathcal{H}}$ -Lipschitz continuous with respect to  $g$  with constants  $\mu, \sigma$  and  $\gamma$ , respectively. Let  $N : H \times H \rightarrow H$  be relaxed monotone in the first argument with constant  $c$  and Lipschitz continuous in the first and second arguments with constants  $\beta$  and  $\xi$ , respectively. If the following condition holds:*

$$\left| \rho - \frac{\xi\sigma(4t\gamma - 1) - c}{\beta^2\mu^2 - \sigma^2\xi^2} \right| < \frac{\sqrt{(\xi\sigma(4t\gamma - 1) - c)^2 - (\beta^2\mu^2 - \sigma^2\xi^2)(1 - 2t\gamma)8t\gamma}}{\beta^2\mu^2 - \sigma^2\xi^2}$$

$$\begin{aligned}
c &> (1 - 4t\gamma)\sigma\xi + \sqrt{(\beta^2\mu^2 - \xi^2\sigma^2)(1 - 2t\gamma)(8t\gamma)} \\
\rho\xi\sigma &< 4t\gamma - 1 \\
\xi\sigma &< \beta\mu,
\end{aligned} \tag{6.2.2}$$

then there exists a set of elements  $u \in H, x \in \tilde{F}(u), y \in \tilde{G}(u), w \in \tilde{A}(u)$  such that  $(u, x, y, w)$  is a solution of (GQCPFM) and

$$g(u_n) \rightarrow g(u), x_n \rightarrow x, y_n \rightarrow y, w_n \rightarrow w \text{ as } n \rightarrow \infty,$$

where  $\{u_n\}, \{x_n\}, \{y_n\}$  and  $\{w_n\}$  are sequences defined in Algorithm 6.2.1.

*Proof.* Since  $m$  is Lipschitz continuous, it follows by Remark 4.1 in [65] that  $\forall x, y, z \in H$ ,

$$\|P_{K(x)}(z) - P_{K(y)}(z)\| \leq 2t\|x - y\|. \tag{6.2.3}$$

From Algorithm 6.2.1, Lemma 1.2.3 and (6.2.3), we have

$$\begin{aligned}
\|g(u_{n+1}) - g(u_n)\| &= \|m(w_n) + P_{K(w_n)}[g(u_n) - m(w_n) - \rho N(x_n, y_n)] \\
&\quad - m(w_{n-1}) - P_{K(w_{n-1})}[g(u_{n-1}) - m(w_{n-1}) - \rho N(x_{n-1}, y_{n-1})]\| \\
&\leq \|m(w_n) - m(w_{n-1})\| + \|P_{K(w_n)}[g(u_n) - m(w_n) - \rho N(x_n, y_n)] \\
&\quad - P_{K(w_n)}[g(u_{n-1}) - m(w_{n-1}) - \rho N(x_{n-1}, y_{n-1})]\| \\
&\quad + \|P_{K(w_n)}[g(u_{n-1}) - m(w_{n-1}) - \rho N(x_{n-1}, y_{n-1})] \\
&\quad - P_{K(w_{n-1})}[g(u_{n-1}) - m(w_{n-1}) - \rho N(x_{n-1}, y_{n-1})]\| \\
&\leq 2\|m(w_n) - m(w_{n-1})\| + \|g(u_n) - g(u_{n-1})\| \\
&\quad - \rho\|N(x_n, y_n) - N(x_{n-1}, y_{n-1})\| + 2t\|w_n - w_{n-1}\| \\
&\leq 2t\|w_n - w_{n-1}\| + \rho\|N(x_{n-1}, y_n) - N(x_{n-1}, y_{n-1})\| \\
&\quad + \|g(u_n) - g(u_{n-1}) - \rho(N(x_n, y_n) - N(x_{n-1}, y_n))\| \\
&\quad + 2t\|w_n - w_{n-1}\|.
\end{aligned}$$

Since  $N(., .)$  is Lipschitz continuous in the second argument and  $m$  is Lipschitz continuous, we have

$$\begin{aligned}
\|g(u_{n+1}) - g(u_n)\| &\leq \|g(u_n) - g(u_{n-1}) - \rho(N(x_n, y_n) - N(x_{n-1}, y_n))\| \\
&\quad + \rho\xi\|y_n - y_{n-1}\| + 4t\|w_n - w_{n-1}\|.
\end{aligned} \tag{6.2.4}$$

By  $\hat{\mathcal{H}}$ -Lipschitz continuity of  $\tilde{F}$  with respect  $g$ , we have

$$\|x_n - x_{n-1}\| \leq (1 + n^{-1})\mu\|g(u_n) - g(u_{n-1})\|. \tag{6.2.5}$$

Since  $N(.,.)$  is relaxed monotone with respect to  $g$  in the first argument, Lipschitz continuous in the first argument and using (6.2.4), we have

$$\begin{aligned}
& \|g(u_n) - g(u_{n-1}) - \rho(N(x_n, y_n) - N(x_{n-1}, y_n))\|^2 \\
&= \|g(u_n) - g(u_{n-1})\|^2 - 2\rho\langle N(x_n, y_n) - N(x_{n-1}, y_n), g(u_n) - g(u_{n-1}) \rangle \\
&\quad + \rho^2 \|N(x_n, y_n) - N(x_{n-1}, y_n)\|^2 \\
&\leq \|g(u_n) - g(u_{n-1})\|^2 + 2\rho c \|g(u_n) - g(u_{n-1})\|^2 + \rho^2 \beta^2 \|x_n - x_{n-1}\|^2 \\
&\leq [1 + 2\rho c + \rho^2 \beta^2 (1 + n^{-1}) \mu^2] \|g(u_n) - g(u_{n-1})\|^2.
\end{aligned} \tag{6.2.6}$$

Further, since  $\tilde{G}$  and  $\tilde{A}$  are  $\hat{\mathcal{H}}$ -Lipschitz continuous with respect to  $g$ , we have

$$\|y_n - y_{n-1}\| \leq (1 + n^{-1}) \hat{\mathcal{H}}(\tilde{G}u_n, \tilde{G}u_{n-1}) \leq (1 + n^{-1}) \sigma \|g(u_n) - g(u_{n-1})\| \tag{6.2.7}$$

and

$$\|w_n - w_{n-1}\| \leq (1 + n^{-1}) \hat{\mathcal{H}}(\tilde{A}u_n, \tilde{A}u_{n-1}) \leq (1 + n^{-1}) \gamma \|g(u_n) - g(u_{n-1})\|. \tag{6.2.8}$$

From (6.2.4) and (6.2.6) - (6.2.8), it follows that

$$\|g(u_{n+1}) - g(u_n)\| \leq \theta_n \|g(u_n) - g(u_{n-1})\|, \tag{6.2.9}$$

where

$$\theta_n = \sqrt{1 + 2\rho c + \rho^2 \beta^2 (1 + n^{-1}) \mu^2} + \rho \xi (1 + n^{-1}) \sigma + 4t(1 + n^{-1}) \gamma$$

Then  $\theta_n \rightarrow \theta$  as  $n \rightarrow \infty$ . From (6.2.2), we have  $\theta < 1$ . Hence  $\theta_n < 1$  for  $n$  sufficiently large. Therefore (6.2.8) implies that  $\{g(u_n)\}$  is a Cauchy sequence in  $g(H)$ . Since  $g(H)$  is closed in  $H$ , there exists  $u \in H$  such that

$$g(u_n) \rightarrow g(u), \text{ as } n \rightarrow \infty.$$

By (6.2.5), (6.2.7) and (6.2.8), it is easy to see that  $\{x_n\}$ ,  $\{y_n\}$  and  $\{w_n\}$  are Cauchy sequences in  $H$ . Since  $H$  is complete, we may let  $x_n \rightarrow x$ ,  $y_n \rightarrow y$ ,  $w_n \rightarrow w$  as  $n \rightarrow \infty$ . Further, we have

$$\begin{aligned}
d(x, \tilde{F}(u)) &= \inf\{\|x - z\| : z \in \tilde{F}(u)\} \\
&\leq \|x - x_n\| + d(x_n, \tilde{F}(u)) \\
&\leq \|x - x_n\| + \hat{H}(\tilde{F}(u_n), \tilde{F}(u)) \\
&\leq \|x - x_n\| + \mu \|g(u_n) - g(u)\| \rightarrow 0
\end{aligned}$$

and hence  $x \in \tilde{F}(u)$ . Similarly  $y \in \tilde{G}(u)$  and  $w \in \tilde{A}(u)$ .

Now we prove that

$$g(u) = m(w) + P_{K(w)}[g(u) - m(w) - \rho N(x, y)]$$

In fact, from Lemma 1.2.3 and (6.2.3), we have

$$\begin{aligned} & \|g(u_{n+1}) - m(w) - P_{K(w)}[g(u) - m(w) - \rho N(x, y)]\| \\ &= \|m(w_n) + P_{K(w_n)}[g(u_n) - m(w_n) - \rho N(x_n, y_n)] \\ &\quad - m(w) - P_{K(w)}[g(u) - m(w) - \rho N(x, y)]\| \\ &\leq \|m(w_n) - m(w)\| + \|P_{K(w_n)}[g(u_n) - m(w_n) - \rho N(x_n, y_n)] \\ &\quad - P_{K(w_n)}[g(u) - m(w) - \rho N(x, y)]\| \\ &\quad + \|P_{K(w_n)}[g(u) - m(w) - \rho N(x, y)] \\ &\quad - P_{K(w)}[g(u) - m(w) - \rho N(x, y)]\| \\ &\leq 2\|m(w_n) - m(w)\| + \|g(u_n) - g(u) - \rho(N(x_n, y_n) - N(x, y))\| \\ &\quad + 2t\|w_n - w\| \\ &\leq \|g(u_n) - g(u)\| + \rho\|(N(x_n, y_n) - N(x, y_n))\| \\ &\quad + \rho\|N(x, y_n) - N(x, y)\| + 4t\|w_n - w\| \\ &\leq \|g(u_n) - g(u)\| + \rho\beta\|x_n - x\| + \rho\xi\|y_n - y\| + 4t\|w_n - w\|. \end{aligned}$$

Since  $g(u_n) \rightarrow g(u)$ ,  $x_n \rightarrow x$ ,  $y_n \rightarrow y$ ,  $w_n \rightarrow w$ , we have

$$g(u_{n+1}) \rightarrow m(w) + P_{K(w)}[g(u) - m(w) - \rho N(x, y)],$$

and therefore

$$g(u) = m(w) + P_{K(w)}[g(u) - m(w) - \rho N(x, y)]$$

This completes the proof. □



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